Randomization in the Josephus Problem

Ioannis Tsokanos
(joint with F. Adiceam, S. Robertson & V. Shirandami)

Universidade Estadual Paulista, Brasil

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Seminários de Probabilidade

The Jewish Wars (by Flavius Josephus)

"Since we all are resolved to die, let us rely on fate to decide the order in which we must kill each other : the first of us that fortune will designate shall fall under a stab from the next one, and thus fate will successively mark the victims and the murderers, exempting us from attempting on our lives with our own hands."

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Josephus Elimination Rule

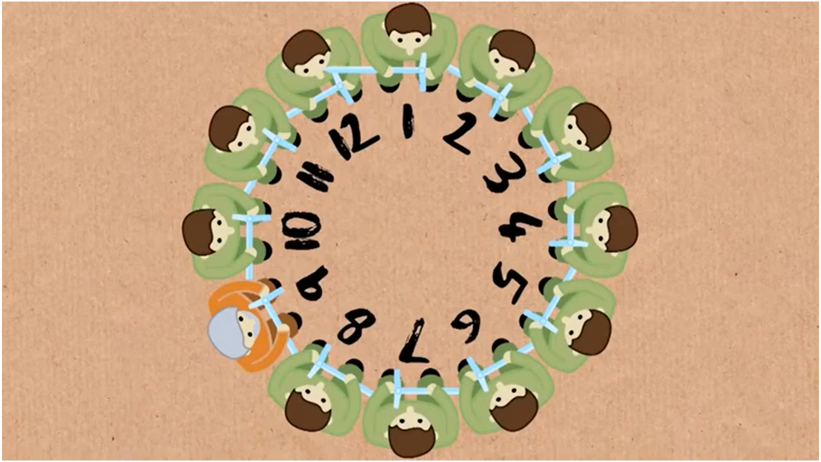


Figure – The media is extracted from the Youtube channel "Numberphile" and its video "The Josephus Problem - Numberphile".

Rule for the Josephus' elimination process (Euler, $XVIII^{th}$ century)

Rule of the Josephus problem :

- N-players, $N \geq 1$, enumerated from 1 to N, stand in a circle.
- Player 1 holds a knife.
- The player holding the knife eliminates his right neighbor and passes the knife onto the next person still alive on his right.

Problem

Given $N \geq 1$, determine the position $a_N \in [\![1, N]\!]$ of the survivor in Josephus game.

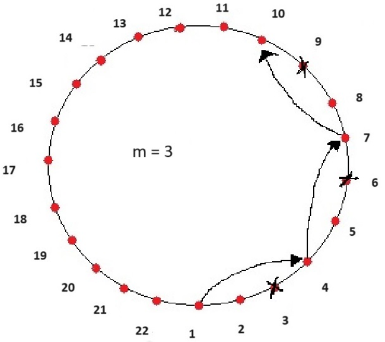
Background

Background : deterministic variations and past literature

Deterministic Variations of the Josephus Problem

Rule for the Generalized Josephus Problem : elimination step $m \in \mathbb{N}$

- N players, enumerated from 1 to N, stand in a circle.
- Player 1 holds a knife.
- The player holding the knife **eliminates the person standing** on the m-th position on his right and **passes the knife** at the player standing at the $m+1$-th position.



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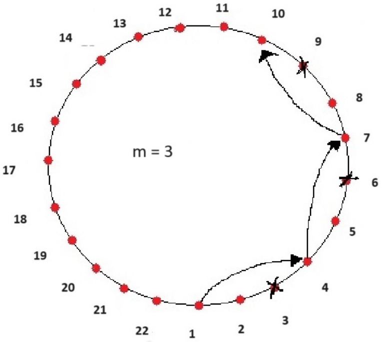

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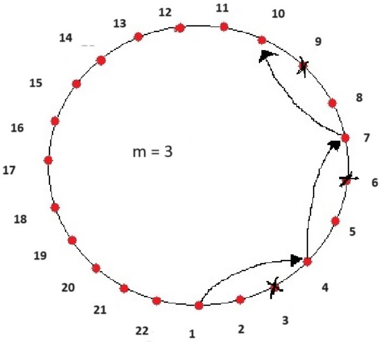

Combinatorial Algorithms and the problem of m-enumerations of Z_N

The Problem of m-enumerations : Let

$$Z_N = \{1, 2, \dots, N\} \quad \text{be an ordered set.}$$

Given $m \in \mathbb{N}$, the m-permutation of Z_N is the ordered set

$$Z_N^{(m)} = \left\{ a_N^{(m)}(1), \dots, a_N^{(m)}(N) \right\},$$

where $a_N^{(m)}(i)$, $1 \leq i \leq N$, is the index of the person eliminated at the i-th step of the Josephus game with elimination step m.

- Given $1 \leq n \leq N$, which $1 \leq i \leq N$ does satisfy $a_N^{(m)}(i) = n$?
- Given $1 \leq i \leq N$, what is the value of $a_N^{(m)}(i)$?

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Past Research on the Josephus Problem

The Past Research :

- Determine the position $a_N^{(m)}(N) \in [\![0, N-1]\!]$ of the survival ;
  - For $m = 2$ (L. Halbeisen & N. Hunberbülher) : close-formula based on a constant $\alpha = 0.8111\ldots$.
  - For $m \geq 3$ (A. Odlyzko & H. Wilf) : close-formula with an error of up to m positions.
- Build algorithms to compute the m-permutation $Z_N^{(m)}$;
  - (L. Lloyd) : algorithm with running time $O(N \cdot \log(m))$.

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The Randomized Version (V1)

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The Probabilistic Elimination Process

Rule for the probabilistic elimination process :

- N players, enumerated from 0 to $N-1$, standing on a unit circle with a regular spacing between them and labeled anticlockwise.
- The 0-th player holds first the knife.
- The player holding the knife :
  1. with probability p eliminates in the same direction as in the previous round and passes the knife in the direction of stabbing.
  2. with probability $1-p$ eliminates in the opposite direction from the previous round and passes the knife in the direction of stabbing.

The Probabilistic Elimination Process : illustration

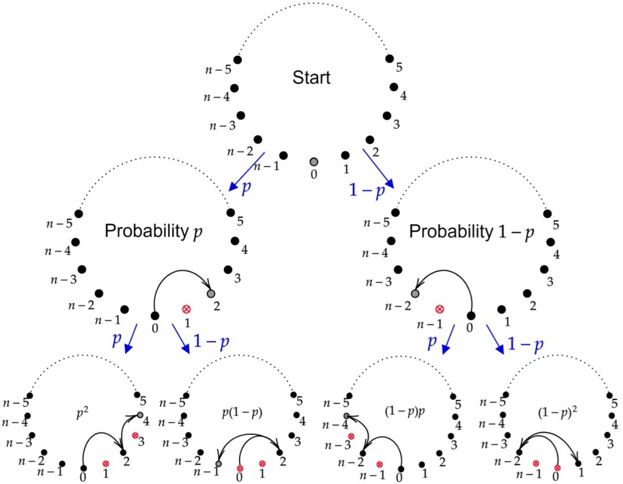

Figure – Illustration of the probabilistic elimination rule over two rounds. The crossed circles indicate the eliminated persons and the dots with grey interior the person who is to make the next move.

The Survival Probabilities and the Probability Measure

- Define

$g_N(n, p) :=$ the survival probability of the person labelled $n \in [\![0, N-1]\!]$.

- Let

$$\mu_N^{(p)} := \sum_{n=0}^{N-1} g_N(n, p) \cdot \delta_{\frac{n}{N}}$$

be a probability measure on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, where δ_x denotes the Dirac mass concentrated at the point $x + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$.

The Problem : determining the limit of the probability distribution

Main Problem (Randomized Version 1)

Determine the limit distribution $\lim_{N\to+\infty} \mu_N^{(p)}$ *of the survivors as the number N of participants tends to infinity.*

In other words, the problem amounts to asking what should be, in the limit, the position on the circle which maximises the chances of survival.

Recall :

$$\mu_N \underset{N\to+\infty}{\longrightarrow} \mu \quad \Leftrightarrow \quad \text{for every } \phi \text{ continuous } \int \phi \cdot d\mu_N \underset{N\to+\infty}{\longrightarrow} \int \phi \cdot d\mu.$$

The Heuristic Approach to the Problem

1. Expectation : the knife moves $p \cdot (1-p)^{-1}$-times towards one direction before moving $p \cdot (1-p)^{-1}$-times to the opposite one.
2. The knife starts an unbiased oscillation around zero.
3. The players around Participant 0 are eliminated first. The players around Participant $N/2$ are eliminated last.
4. The smaller $p \in (0,1) \leftrightarrow$ More changes of the direction $\leftrightarrow$ More intense oscillation.

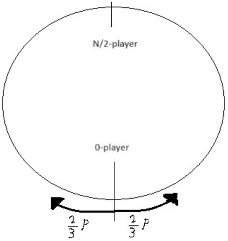

Figure – The unbiased oscillation of the knife under the probabilistic elimination process.

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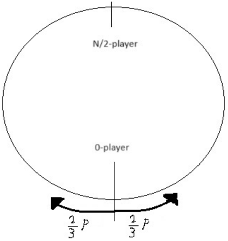

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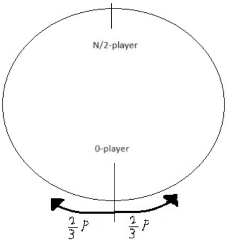

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Numerical Data : computer simulations

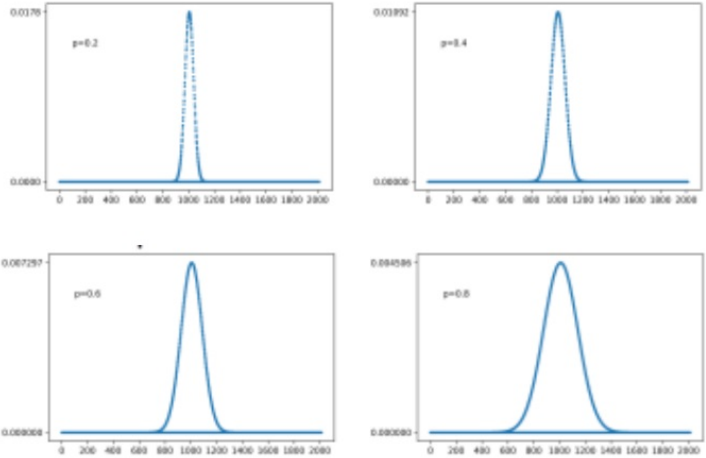

Figure – Computer simulations for a game with 2000 players and the values of p varying in the interval $(0, 1)$.

The Main Result

Theorem (F. Adiceam, S. Robertson, V. Shirandami & I. T.)

Assume that $p \in (0,1)$. Then, the sequence of measures $\left(\mu_N^{(p)}\right)_{N\geq 1}$ admits a weak limit $\mu^{(p)}$ supported in $\mathbb{R}/\mathbb{Z}$.

(1) ***General case (linear combination) :*** *When $p \in (0,1)$, the limit measure $\mu^{(p)}$ is a convex combination of Dirac masses concentrated at the origin and at the point $1/2$; that is,*

$$\mu^{(p)} = (1-c_p)\cdot\delta_0 + c_p\cdot\delta_{1/2}, \quad \text{for some } c_p \in [0,1]\,.$$

(2) ***Middle interval (specific position) :*** *When $p \in (1/3, 2/3)$, the limit measure $\mu^{(p)}$ is the Dirac mass concentrated at the point $1/2$; that is,*

$$\mu^{(p)} = \delta_{1/2} \quad \Leftrightarrow \quad c_p = 1\,.$$

The Main Result : unbiased case

Theorem (F. Adiceam, S. Robertson, V. Shirandami & I. T.)

(3) ***Unbiased case (Centeral Limit Theorem) :*** *When $p = 1/2$, if $(X_N)_{N\geq 1}$ is a sequence of random variables drawn successively and independently, each according to the probability measure $\mu_N^{(1/2)}$, then*

$$\frac{1}{S_L} \cdot \sum_{N=1}^{L} \left(X_N - \frac{1}{2} \right) \underset{L\to+\infty}{\to} \mathcal{N}(0,1),$$

with

$$S_L = \sqrt{\sum_{N=1}^{L} \mathbb{V}\left(X_N\right)} \quad \text{and} \quad S_L \asymp \sqrt{\ln(L)},$$

where $\mathbb{V}\left(X_N\right)$ is the variance of the random variable X_N.

Proof of the Main Result : the analytic operator

$$\phi \in \mathcal{C}^0(\mathbb{T}) \quad \leftrightarrow \quad \phi \in \mathcal{C}^0_P([0,1]) : \text{boundary condition } \ \phi(0) = \phi(1).$$

The analytic operator :

$$\begin{aligned} J_N^{(p)} : \phi \in \mathcal{C}^0_P([0,1]) \mapsto J_N^{(p)}[\phi] &= \int_{\mathbb{R}/\mathbb{Z}} \phi \cdot d\mu_N^{(p)} \\ &= \sum_{n=0}^{N} \phi\left(\frac{n}{N}\right) \cdot g_N(n,p) \end{aligned}$$

Main Problem (Reformulated)

Estimate the limit

$$\lim_{N\to+\infty} J_N^{(p)}[\phi] \quad \textit{for every } \phi \in \mathcal{C}^0_P([0,1]).$$

Proof of the Main Result : Weierstrass theorem

Odd around $1/2$	**Even** around $1/2$
$\phi(x) = -\phi(1-x)$	$\phi(x) = \phi(1-x)$

1. Every function ϕ can be decomposed in odd and even parts :

$$\phi \ = \ \phi_O + \phi_E; \quad \phi_O(x) = \frac{\phi(x) - \phi(1-x)}{2} \ \text{and} \ \phi_E(x) = \frac{\phi(x) + \phi(1-x)}{2}.$$

2. *Weierstrass Approximation Theorem :* every $\phi \in \mathcal{C}^0_P([0,1])$ can be approximated arbitrary well by polynomials.

3. The **even functions** can be decomposed as a linear combination of the $2k$-movements $\phi_{2k} : [0,1] \mapsto \mathbb{R}$, where

$$\phi_{2k}(x) \ = \ \left(x - \frac{1}{2}\right)^{2k}.$$

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Proof of the Main Result : the recursive relations

Proposition (Recursion Relations for the Probability of Survival)

Let $N \geq 3$ and $p \in (0,1)$. Whenever $N \geq 4$, the probability vector $(g_N(n,p))_{0 \leq n \leq N-1}$ meets the recurrence relation

$$g_N(n,p) \;=\; \begin{cases} g_{N-1}(-1,p), & \text{if } n \equiv 0 \pmod N \\ (1-p)\cdot g_{N-1}(-2,p), & \text{if } n \equiv 1 \pmod N \\ p \cdot g_{N-1}(-2,p), & \text{if } n \equiv -1 \pmod N \\ p \cdot g_{N-1}(n-2,p) + (1-p)\cdot g_{N-1}(N-n-2,p), & \text{otherwise} \end{cases},$$

with the base $(g_3(0,p), g_3(1,p), g_3(2,p)) = (0, 1-p, p)$.

Proof of the Main Result : the limiting behavior of the random process

The recursive relations can be used to show the following two lemmas.

Lemma (Odd Functions)

Let $\phi_O \in \mathcal{C}_P^0([0,1])$ be an odd function with periodic boundary conditions. Then, there exists a constant $C(\phi_O, p) > 0$ depending only on ϕ_O and $p \in (0,1)$ such that

$$\left| J_N^{(p)}[\phi_O] \right| \leq \frac{C(\phi_O, p)}{N}.$$

Lemma (2k-moments)

Let $2k \in 2\mathbb{N}$ be an even non-negative integer. Then, for every $N \geq 4$, it holds that

$$J_N^{(p)}[\phi_{2k}] - J_{N-1}^{(p)}[\phi_{2k}] = \frac{2k}{N} \cdot J_{N-1}^{(p)}[\phi_{2k}] + O_{\phi,p}\left(\frac{1}{N^2}\right).$$

Furthermore, the sequence $\left(J_N^{(p)}[\phi_{2k}]\right)_{N \geq 1}$ converges.

Proof of the Main Result : weak convergence of the probability measures

1. The linear operator

$$J^{(p)} : \phi \mapsto \lim_{N \to +\infty} J_N^{(p)} [\phi] , \quad \phi \in \mathcal{C}_P^0 ([0, 1])$$

   is **well-defined and positive** (i.e. $J^{(p)} [\phi] \geq 0$ whenever $\phi \geq 0$).

2. *Riesz Representation Theorem :* there exists a (probability) measure $\mu^{(p)}$ of finite mass such that

$$J^{(p)} [\phi] = \int_{\mathbb{R}/\mathbb{Z}} \phi \cdot d\mu^{(p)}.$$

3. Therefore,

$$\mu_N^{(p)} \xrightarrow[N \to +\infty]{} \mu^{(p)}.$$

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3. Therefore,

$$\mu_N^{(p)} \underset{N \to +\infty}{\longrightarrow} \mu^{(p)}.$$

Proof of the Main Result : Cases 1 & 2

- **Case** $p \in (0, 1)$ **:** An analytic argument yields that

$$\mu^{(p)} = (1 - c_p) \cdot \delta_0 + c_p \cdot \delta_{1/2}, \quad \text{for some} \;\; c_p \in [0, 1].$$

- **Case** $p \in (1/3, 2/3)$ **:**
  - Exponential Upper Bound :

$$g_N(n, p) \ll_p \frac{\beta^{|n|}}{\gamma^N}, \quad \text{for some} \;\; \beta > 0 \text{ and } \gamma > 1.$$

  - $\Rightarrow$ “no dependence on 0”, i.e.

$$\mu^{(p)} = \delta_{1/2}.$$

Proof of the Main Result : unbiased case

Case $p = 1/2$ **:** To prove

$$\frac{1}{S_L} \cdot \sum_{N=1}^{L} \left(X_N - \frac{1}{2} \right) \underset{L \to +\infty}{\longrightarrow} \mathcal{N}(0,1)$$

it suffices to show that

$$\frac{1}{S_L} \cdot \sum_{N=1}^{L} \left(X_N - \mathbb{E}\left(X_N\right)\right) \underset{L \to +\infty}{\longrightarrow} \mathcal{N}(0,1) \quad \text{and} \quad \frac{1}{S_L} \cdot \sum_{N=1}^{L} \left(\mathbb{E}\left(x_N\right) - \frac{1}{2} \right) \underset{L \to +\infty}{\longrightarrow} 0.$$

Proof of the Main Result : Unbiased Case

- **Lyapunov Condition :** If

$$\lim_{N \to +\infty} \left(\frac{1}{S_L^3} \cdot \sum_{N=1}^{L} \mathbb{E}\left(\left| X_N - \mathbb{E}(X_N) \right|^3 \right) \right) = 0,$$

then

$$\frac{1}{S_L} \cdot \sum_{N=1}^{L} (X_N - \mathbb{E}(X_N)) \underset{L \to +\infty}{\longrightarrow} \mathcal{N}(0,1).$$

- **Slutsky Condition :** If $\sum_{N=1}^{+\infty} (\mathbb{E}(X_N) - 1/2) < +\infty$ and $S_L \underset{L \to +\infty}{\longrightarrow} +\infty$, then

$$\frac{1}{S_L} \cdot \sum_{N=1}^{L} \left(\mathbb{E}(x_N) - \frac{1}{2} \right) \underset{L \to +\infty}{\longrightarrow} 0.$$

Proof of the Main Result : Unbiased Case

- Strong Exponential Upper Bound :

$$g_N\left(n, \frac{1}{2}\right) \ll_p \frac{\alpha^{2\cdot(1+\epsilon)n}}{\alpha^N}, \quad \text{for some } \alpha = \alpha(\epsilon, p) > 1.$$

- $\Rightarrow$ Sharp estimations of the k-th moments.

- $\Rightarrow$ Lyapunov and Slutsky conditions.

Open Problems and Conjectures

In view of the main result, the previously mentioned heuristic combined with the numerical data gathered from computer simulations indicate that :

Conjecture (Randomized Version 1)

1. *For every* $p \in (0, 1/3] \cup [2/3, 1)$*, it holds that*

$$\mu_N^{(p)} \underset{N \to +\infty}{\longrightarrow} \delta_{1/2}.$$

2. *For (at least) every* $p \in (0, 1/2]$*, the rate of convergence can be given in the form of a Central Limit Theorem.*

The Alternative Randomized Version (V2)

The Alternative Randomized Version (V2)

Randomization in the Josephus Problem : version 2

(Alternative) Rule for the probabilistic elimination process :

- N players, enumerated from 0 to $N-1$, standing on a unit circle with a regular spacing between them and labeled anticlockwise.
- The 0-th player holds first the knife.
- The player holding the knife :
  1. with probability p eliminates the player standing on his right and passes the knife in the same direction.
  2. with probability $1-p$ eliminates the player standing on his left and passes the knife in the same direction.

The (Alternative) Probabilistic Elimination Process : illustration

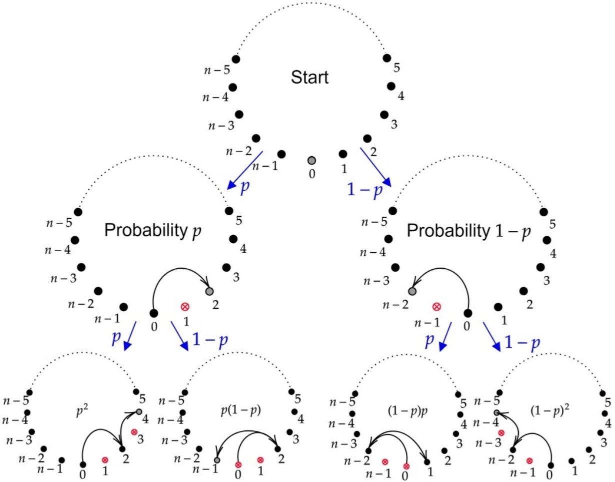

Figure – Illustration of the probabilistic process with the alternative elimination rule over two rounds. The crossed circles indicate the eliminated persons and the dots with grey interior the person who is to make the next move.

The Survival Probabilities and the Probability Measure (Alternative Version)

- Define

$$f_N(n,p) := \quad \text{the survival probability of the person labelled } n \in [\![0, N-1]\!] \,.$$

- Let

$$\nu_N^{(p)} := \sum_{n=0}^{N-1} f_N(n,p) \cdot \delta_{\frac{n}{N}}$$

be a probability distribution on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

The Problem : determining the limit of the probability distribution

Main Problem (Randomized Version 2)

Determine the limit distribution $\lim_{N \to +\infty} \nu_N^{(p)}$ *of the survivors as the number N of participants tends to infinity.*

In other words, the problem amounts to asking what should be, in the limit, the position on the circle which maximises the chances of survival.

The Heuristic : an intuitive approach to the (alternative) problem

1. Expectation : the knife moves p-times towards one direction before moving $(1-p)$-times to the opposite one.
2. If $p \in (1/3, 2/3)$, then the knife knife starts a biased oscillation around the 0-player. This leads to a convergence behavior.
3. If $p \in (0, 1/3) \cup (2/3, 1)$, then the knife eventually starts circling. This seems to lead to a divergence behavior.

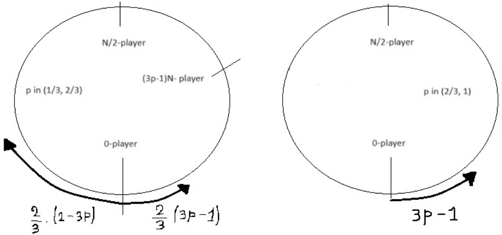

Figure – The biased oscillation/circling of the knife under the (alternative) probabilistic elimination process.

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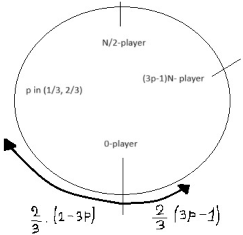

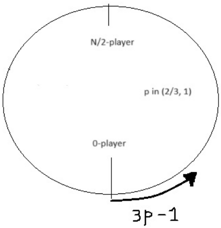

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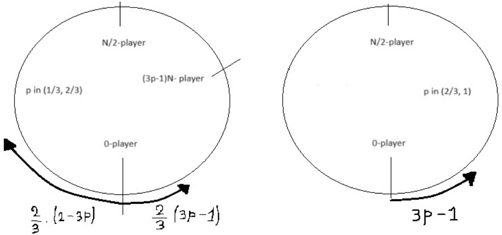

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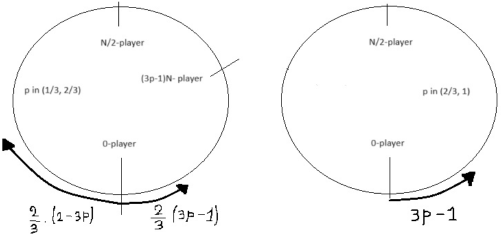

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The Heuristic : computer simulations (alternative version)

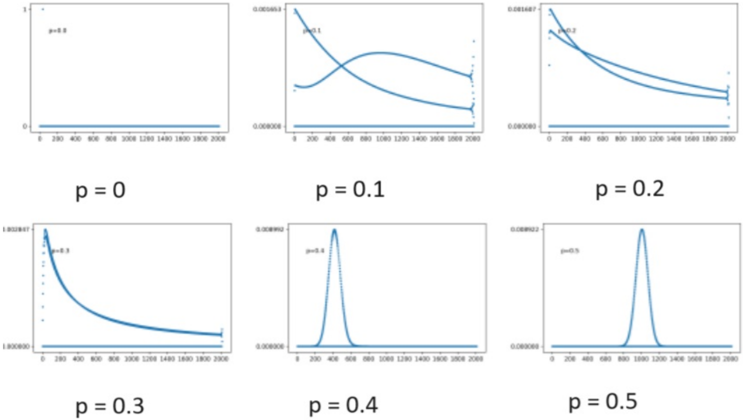

Figure – Computer simulations for a game with 2000 players and the values of p varying in the interval $(0, 1)$.

The Main Result

Theorem (F. Adiceam, S. Robertson, V. Shirandami & I. T.)

Assume that $p \in (1/3, 2/3)$. Then, the sequence of probability measures $\left(\nu_N^{(p)}\right)_{N\geq 1}$ admits a weak limit $\nu^{(p)}$ supported in $\mathbb{R}/\mathbb{Z}$, namely,

$$\nu_N^{(p)} \underset{N\to+\infty}{\longrightarrow} \delta_{3p-1}.$$

Moreover, in the unbiased case $p = 1/2$, the convergence to $\delta_{1/2}$ can be expressed in the form of a Central Limit Theorem.

The Recursion Relations and an Upper Bound for the Survival Probabilities

Proposition ((Alternative) Recursion Relations for the Probability of Survival)

Let $N \geq 3$ and $p \in (0,1)$. Whenever $N \geq 4$, the probability vector $(g_N(n,p))_{0\leq n\leq N-1}$ meets the recurrence relation

$$f_N(n,p) = \begin{cases} p \cdot f_{N-1}(-1,p) + (1-p)\cdot f_{N-1}(1,p), & \text{if } n \equiv 0 \pmod{N} \\ (1-p)\cdot f_{N-1}(2,p), & \text{if } n \equiv 1 \pmod{N} \\ p \cdot f_{N-1}(-2,p), & \text{if } n \equiv -1 \pmod{N} \\ p \cdot f_{N-1}(n-2,p) + (1-p)\cdot f_{N-1}(n+1,p), & \text{otherwise} \end{cases},$$

with the base $(f_3(0,p), f_3(1,p), f_3(2,p)) = (0, 1-p, p)$.

Proposition (Upper Bounds for the Survival Probabilities)

Fix $p \in (1/3, 2/3)$ and $\epsilon > 0$. There exists constant $\alpha = \alpha(\epsilon, p) > 1$ such that for all $N \geq 1$ and $0 \leq n \leq N-1$,

$$f_N(n,p) \ll_{\epsilon,p} \frac{\alpha^{\frac{(1+\epsilon)}{(3p-1)}n}}{\alpha^N} \quad \text{and} \quad f_N(n,1-p) \ll_{\epsilon,p} \frac{\alpha^{\frac{(1+\epsilon)}{(2-3p)}n}}{\alpha^N}.$$

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Open Problems and Conjectures

Conjecture (Randomized Version 2)

Let $p \in (0, 1/3) \cup (2/3, 1)$. Then, the sequence of probability measures $\left(\nu_N^{(p)}\right)_{N \geq 1}$ diverges.

General Rule for the Probabilistic Elimination Process

General Rule for the Probabilistic Elimination Process :

- N players, enumerated from 0 to $N-1$, standing on a unit circle with a regular spacing between them and labeled anticlockwise.
- The 0-th player holds first the knife.
- The player holding the knife :
  1. with probability p eliminates the player standing on his right.
  2. with probability $1-p$ eliminates the player standing on his left.
  3. After eliminating, with probability q passes the knife on his right side and with probability $1-q$ passes the knife on his left side.

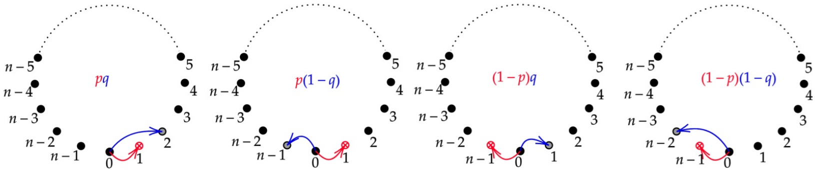

Figure – Illustration of the probabilistic process with the general rule.

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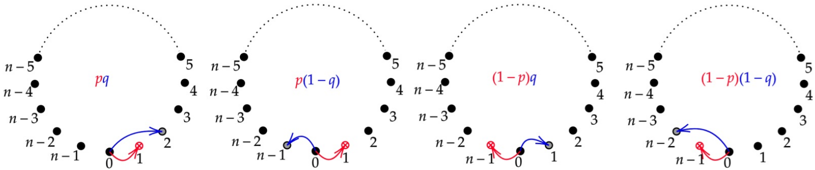

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General Rule for the Probabilistic Elimination Process : simulations

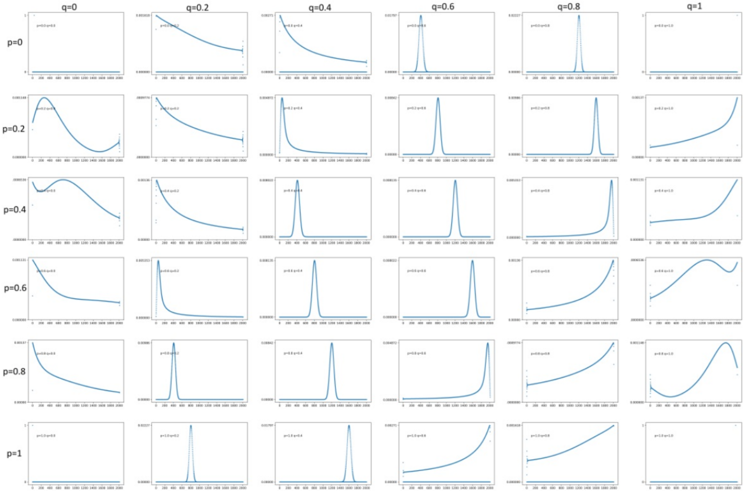

Figure – Computer simulations for a game with 2000 players with the values of the parameters p and q varying in $[0, 1]$.

Thank you for watching !

Figure – Sunset in the lake of Sao Jose do Rio Preto.