Results on the contact process on interchange process

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Introduction

Growth processes

Models for growth of something over time

(something: infection, population, information, forest fire, etc.)

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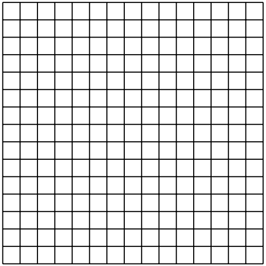
Harris Contact Process: spread of infection on a connected graph $G = (V, E)$ with bounded degree



Growth processes

Harris Contact Process has a static environment.

- One particle at each site of V.
- Infection spreads to neighbors always at the same rate.
- Both remain the same forever.

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Dynamic environment:

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We discuss two models with dynamic environments:

- Contact Process with Dynamic Edges (CPDE)
  Linker and Remenik (2020)
- Contact Process on Interchange Process (CPIP)
  Hilário, U., Valesin, Vares (2025+)

Contact Process on $\mathbb{Z}^d$

Markov process $\{\zeta_t\}_{t\geq 0}$ with values on $\{Ⓗ, ⓘ\}^V$:

- $\zeta_t(x) = ⓘ$ means x is infected at time t
- $\zeta_t(x) = Ⓗ$ means x is healthy at time t

Competing states time evolution:

- Infected $\longrightarrow$ healthy: rate 1
- Healthy $\longrightarrow$ infected: rate $\lambda \times$ (# infected neighbors)

Initial states: for $A \subset V$,

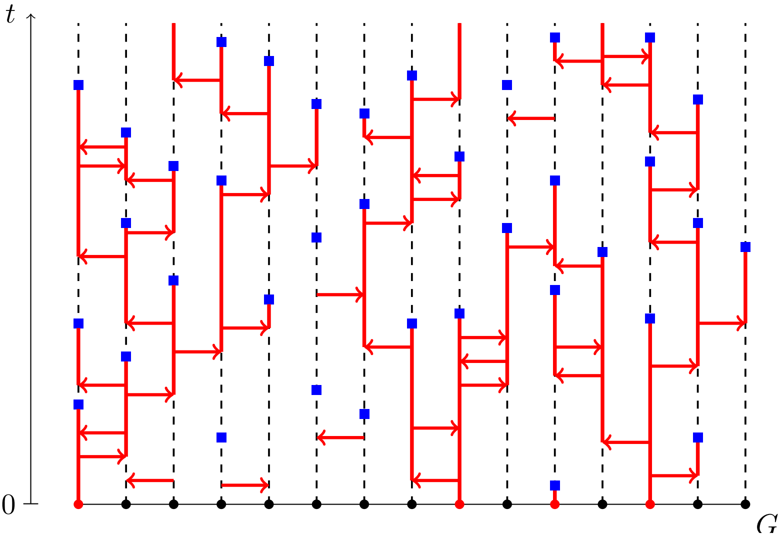
$$(\zeta_t^A)_{t\geq 0} \text{ is the CP started from } \zeta_0^A(x) = ⓘ \iff x \in A.$$

Absorbing states: $\zeta_t \equiv Ⓗ$.

Phase transition:

$$\lambda_c := \inf\{\lambda > 0; \mathbb{P}_\lambda(\forall t > 0\ \exists x\colon \zeta_t^{\{0\}}(x) = ⓘ) > 0\} \in (0, \infty)$$

Graphical Representation (CP)



■ Cures: $\{\mathcal{R}_x\}$ rate 1; $\rightarrow$ Transmissions: $\{\mathcal{T}_e\}$ rate λ;

Contact Process with Dynamic Edges

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Linker and Remenik (2020)

Graph. dynamic environment process $\zeta_t \in \{ⓗ, ⓘ\}^E$.

Cures. Poisson of rate 1;

Infections. Poisson of rate λ.

Evolution of the model

- Edge density: $p \in [0,1]$;

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- Updates: speed $\mathsf{v} > 0$

$$0 \longrightarrow 1 \quad \text{at rate} \quad \mathsf{v}p,$$
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- At any time: $\zeta_t \overset{d}{=}$ Bernoulli(p) bond percolation;
- Transmissions: Poisson of rate λ, but only when edge is open.

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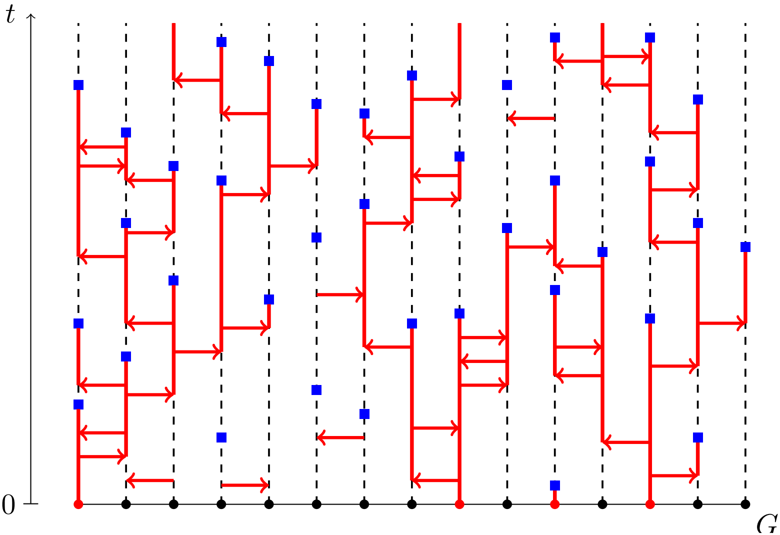

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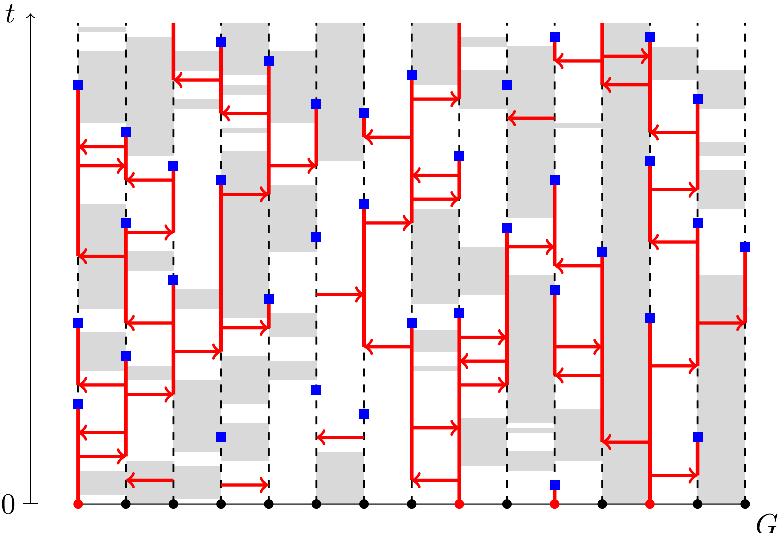

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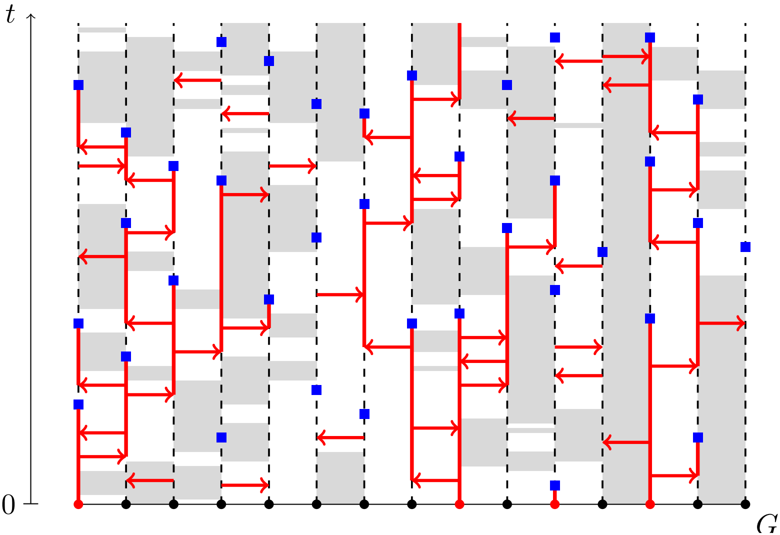

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Results for CPDE on $\mathbb{Z}$

Linker and Remenik (2020):
Extinction/survival when $\mathsf{v} \to \infty$ or $\mathsf{v} \to 0$.

$\mathsf{v} \to \infty$: Mean-field behavior

- '$\mathsf{v} = \infty$' can be seen as a thinning of Ppps:

$$\text{Transmission marks } (\mathcal{T}_e) \overset{d}{=} \text{PPP}(\lambda);$$

$$\text{Allowed transmissions} \overset{d}{=} \text{PPP}(p\lambda).$$

- Heuristically, ζ_t is Contact Process with infection rate $p\lambda$.

Defining

$$\lambda_0(\mathsf{v}, p) := \inf\left\{\lambda > 0;\ \mathbb{P}_{\mathsf{v},p,\lambda}\left(\forall t > 0\ \exists x : \zeta_t^{\{0\}}(x) = \text{ⓘ}\right) > 0\right\}.$$

Theorem. For all $p \in (0, 1]$, $\lim_{\mathsf{v}\to\infty} \lambda_0(\mathsf{v}, p) = \lambda_c(1)/p$.

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$\mathsf{v} \to 0$: Static behavior

- '$\mathsf{v} = 0$' can be seen as being static:
  Initial state of edges never change.
- Heuristically, ζ_t is Contact Process with infection rate λ on a bond percolation configuration.

Theorem. For all $p \in [0, 1)$, $\lim_{\mathsf{v} \to 0} \lambda_0(\mathsf{v}, p) = \infty$.

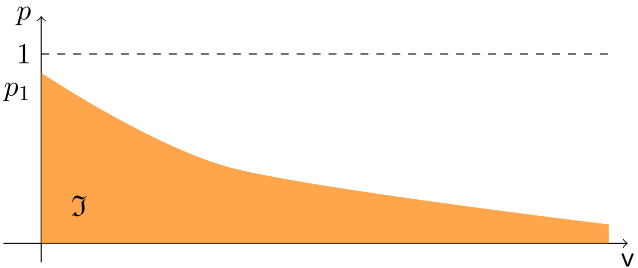
Results for CPDE on $\mathbb{Z}$

Linker and Remenik (2020):

Interesting feature: **immunity region**

- Define $\mathfrak{I} = \{(\mathsf{v}, p) \colon \lambda_0(\mathsf{v}, p) = \infty\}$.

  $(\mathsf{v}, p) \in \mathfrak{I}$ means even when $\lambda = \infty$ there is extinction.



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Other graphs:

- Almost all results are proven for infinite vertex-transitive graphs with bounded degree.
- Exception:

  Theorem. On $\mathbb{Z}$, for all $p \in [0,1)$ we have $\lim_{\mathsf{v}\to 0} \lambda_0(\mathsf{v}, p) = \infty$.

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$p < 1$ and v sufficiently small $\implies \mathbb{P}_{\mathsf{v},p,\lambda}(\tau^{\{0\}} = \infty) = 0.$

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$p < 1$ and v sufficiently small	$\Longrightarrow$	$\mathbb{P}_{\mathsf{v},p,\lambda}(\tau^{\{0\}} = \infty) = 0.$
subcritical and frozen environment	$\Longrightarrow$	infection dies

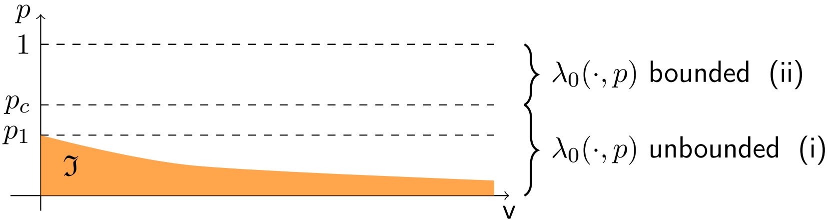
Results for CPDE on $\mathbb{Z}^d$

Theorem 1 (Hilário, U., Valesin, Vares (2022))

For $\mathbb{Z}^d$ with $d \geq 2$:

(i) For all $p < p_c(d)$, $\lim_{\mathsf{v} \to 0} \lambda_0(\mathsf{v}, p) = \infty$

(ii) For all $p > p_c(d)$, $\sup_{\mathsf{v} > 0} \lambda_0(\mathsf{v}, p) < \infty$.



Results for CPDE

Remarks:

- We don't know much about $\mathsf{v} \mapsto \lambda_0(\mathsf{v}, p)$.
  Only: $\mathsf{v} \mapsto \frac{1}{\mathsf{v}} \lambda_0(\mathsf{v}, p)$ is non-increasing.
- Theorem 1 (ii) shows $p_1 \leq p_c(d)$.
  For $d = 1$ it holds $p_1 < 1 = p_c(1)$.
- **Open:** What happens at $p = p_c(d)$ for $d \geq 2$?

Related works:

- Broman (2007)
- Steif and Warfheimer (2008)
- Seiler and Sturm (2023)

Contact Process on Interchange Process

Contact process on Interchange Process

State space. $\zeta_t \in \{0, ⓗ, ⓘ\}^{\mathbb{Z}^d}$.

$0 \leftrightarrow$ no particle, ⓗ $\leftrightarrow$ healthy particle, ⓘ $\leftrightarrow$ infected particle;

Graphical Construction. Has transmissions, cures and *jumps*.

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Evolution of the model

- Initial environment: proportion of particles $p \in (0, 1]$
  - $\zeta_0(0) =$ ⓘ;
  - For other $x \in \mathbb{Z}^d$: independently,

$$\mathbb{P}(\zeta_0(x) = \text{ⓗ}) = p \quad \text{and} \quad \mathbb{P}(\zeta_0(x) = 0) = 1 - p$$

- Jumps: speed $\mathsf{v} > 0$
  - Interchange: jump switches any two states;
  - $\Longrightarrow$ particles perform Simple Random Walks;
- At any time: particles distributed as Bernoulli percolation;
- Particles carry the infection;

Contact process on Interchange Process

Let $\mathbb{P}_{\lambda,\mathsf{v},p}$ denote the law of $(\zeta_t)_{t\geq 0}$. Define

$$\Theta(\lambda,\mathsf{v},p) := \mathbb{P}_{\lambda,\mathsf{v},p}(\text{for all } t \text{ there exists } x \text{ such that } \zeta_t(x) = \text{ⓘ}),$$
$$\lambda_c(\mathsf{v},p) := \inf\{\lambda > 0 : \Theta(\lambda,\mathsf{v},p) > 0\}.$$

Question. What is the behavior of $\lambda_c(\mathsf{v},p)$ as $\mathsf{v} \to 0$ and $\mathsf{v} \to \infty$?

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Special case: $p = 1$

- Reduces to the known contact + stirring
- Behavior as $\mathsf{v}\to\infty$ studied since late 1980s: (De Masi, Ferrari, Lebowitz; Durrett, Neuhauser; Katori; Konno; Bransom et al; Berezin, Mytnik; Levit, Valesin; ...)
- Critical parameter: comparison with branching process
  - Birth with rate $2d\lambda$ and death with rate 1.
  - $\implies \lambda_c \sim \frac{1}{2d}$ as $\mathsf{v}\to\infty$.
- Higher order approximations in v for λ_c are known.

Results for CPIP

Case v $\to \infty$:

Mean-field heuristics: similar to Branching Process

- When particle tries to infect: independent neighborhood
  $\Longrightarrow$ birth with rate $2dp\lambda$ and death with rate 1.

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Theorem 2 (Hilário, U., Valesin, Vares (2025+))

For CPIP on $\mathbb{Z}^d$ with $d \geq 1$ with $p \in (0, 1]$:

$$\lim_{\mathsf{v}\to\infty} \lambda_c(p, \mathsf{v}) = \frac{1}{2dp}.$$

In other words we have:

Extinction. $2dp\lambda < 1 \quad \implies \quad \Theta(\lambda, \mathsf{v}, p) = 0, \ \forall \mathsf{v} \geq \mathsf{v}_0.$

Survival. $2dp\lambda > 1 \quad \implies \quad \Theta(\lambda, \mathsf{v}, p) > 0, \ \forall \mathsf{v} \geq \mathsf{v}_1.$

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Case $v \to 0$: As the previous heuristics suggests:

subcritical and frozen environment $\Longrightarrow$ infection dies

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Theorem 3 (Hilário, U., Valesin, Vares (2025+))

For CPIP on $\mathbb{Z}^d$ with $p < p_c(\mathbb{Z}^d, \mathsf{site})$ we have

$$\lim_{\mathsf{v}\downarrow 0} \lambda_c(p, \mathsf{v}) = \infty.$$

Remarks.

- We have not addressed survival part.
- Theorem 3 also holds when Interchange Process is replaced by Exclusion Process.
- **Open:** Is there an immunity region for $\mathsf{v} > 0$?

Proof Ideas

Renormalization

Multi-scale renormalization. For $N \in \mathbb{N}$, define scale N boxes:

$$\mathcal{Q}_N := [-L_N, L_N]^d \times [0, h_N].$$

Cascading events.

- Define events $\{\mathcal{Q}_N \text{ is bad}\}$.
- $\mathcal{Q}_N$ is bad $\implies$ two far away copies of $\mathcal{Q}_{N-1}$ are bad.

Decoupling. $\mathbb{P}(\mathcal{Q}_{N+1} \text{ is bad}) \leq \mathbb{P}(\mathcal{Q}_N \text{ is bad})^2 + \text{small error}.$

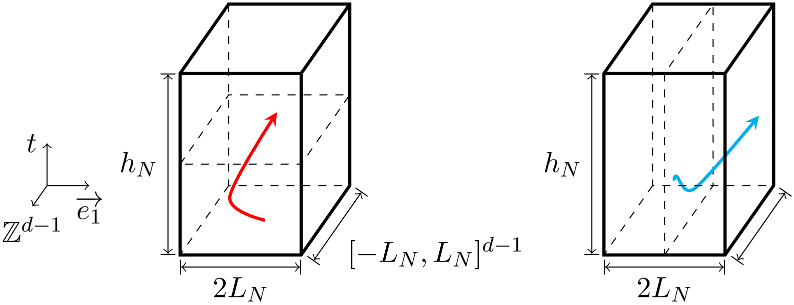
$\mathcal{Q}_N(x_1, t_1)$ and $\mathcal{Q}_N(x_2, t_2)$ are far away $\implies$ $\mathcal{Q}_N(x_1, t_1)$ and $\mathcal{Q}_N(x_2, t_2)$ are close to independent

Trigger. Choose parameters so that $\mathbb{P}(\mathcal{Q}_0 \text{ is bad})$ is small.

Induction argument $\implies$ $\mathbb{P}(\mathcal{Q}_N \text{ is bad}) \to 0.$

Cascading events.

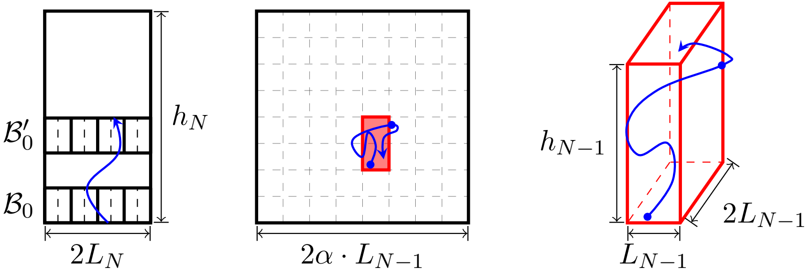
- $L_N = \alpha^N \cdot L_0$ and $h_N = \alpha^N \cdot h_0$;
- $\mathcal{Q}_N(x,t) = (x,t) + \mathcal{Q}_N$ is bad if it is half-crossed.



$$\mathbb{P}(\tau^0 = \infty) \leq \mathbb{P}(\mathcal{Q}_N \text{ is half-crossed}) \xrightarrow{N\to\infty} 0.$$

Cascading events for Extinction

half-crossing of $\mathcal{Q}_N \quad \Longrightarrow \quad$ Two half-crossings



$$H(Q_N) := \{\mathcal{Q}_N \text{ half-crossed}\} \subset \bigcup_{j=0}^{2d} \Big(\bigcup_{(B,B') \in \mathcal{B}_j \times \mathcal{B}'_j} H(B) \cap H(B') \Big).$$

Entropy: number of pairs of boxes is $C(d, \alpha)$ (independent of N).

Decoupling – Discrepancy for IP

Interchange flow:

For $x \in \mathbb{Z}^d$ and $s \geq 0$ define $t \mapsto \Phi(x,s,t)$ such that

- $\Phi(x,s,s) = x$ for every x and for $t > s$;
- $\Phi(x,s,t-) = y,\ t \in \mathcal{J}_{\{y,z\}} \implies \quad \Phi(x,s,t) = z$;
- $\Phi(x,s,t-) = y,\ t \notin \cup_{z\sim y}\mathcal{J}_{\{y,z\}} \implies \quad \Phi(x,s,t) = y$.
- For $s > t$, $\Phi(\cdot,s,t)$ is the inverse function.

Flow gives the trajectories of IP: $\quad \xi_t(x) = \xi_0(\Phi(x,t,0))$.

Definition. (Discrepancy probability for the IP)
Let Φ (rate v $= 1$) be the flow. For $\ell < L \in \mathbb{N}$ and $t > 0$,

$$\mathrm{discr}^{\mathrm{ip}}(\ell,L,t) := \mathbb{P}\left(\begin{array}{l} \exists x \in \partial B_0(L), \\ 0 \leq s < s' \leq t \end{array} \;:\; \Phi(x,s,s') \in \partial B_0(\ell) \right)$$

Decoupling – Discrepancy for CPIP

Containment flow:

For $x \in \mathbb{Z}^d$ and $t \geq s \geq 0$ define $t \mapsto \Psi(x,s,t) \subset \mathbb{Z}^d$:

$$\Psi(x,s,t) := \begin{array}{l} \text{infected set at time } t \text{ of a CPIP that ignores cure marks} \\ \text{started from } \zeta_s(x) = \text{ⓘ and ⓗ otherwise} \end{array}$$

Definition. (Discrepancy probability for the CPIP)

Let Ψ be the containment flow. For $\ell < L \in \mathbb{N}$ and $t > 0$,

$$\begin{aligned} &\mathrm{discr}^{\mathrm{icp}}_{\mathsf{v},\lambda}(\ell, L, t) \\ &:= \mathbb{P}\left(\begin{array}{c} \text{there exist } x \in \partial B_0(L),\ y \in \partial B_0(\ell) \text{ and } s, s' \in [0,t] \\ \text{with } 0 \leq s < s' \leq t \text{ such that } y \in \Psi(x,s,s') \end{array} \right). \end{aligned}$$

Decoupling – Discrepancy estimates

Lemma. (Discrepancy probability estimate for the IP)
Consider Φ with rate $\mathsf{v} = 1$. For $\ell < L \in \mathbb{N}$ and $t > 0$,

$$\mathrm{discr}^{\mathrm{ip}}(\ell, L, t) \leq 16ed^3 tL^{d-1} \exp\left\{-(L-\ell)\log\left(1+\frac{L-\ell}{2t}\right)\right\}.$$

Proof. SRW estimates.

Lemma. (Discrepancy probability estimate for the CPIP)
For any $\mathsf{v} > 0$, $\lambda > 0$, $\ell, L \in \mathbb{N}$ with $\ell < L$ and $t \geq 1$, we have

$$\mathrm{discr}^{\mathrm{icp}}_{\mathsf{v},\lambda}(\ell, L, t)$$
$$\leq c\max(\mathsf{v}^2, 1)(\ell L)^{d-1} \cdot te^{8d\lambda t} \cdot \exp\left\{-\frac{1}{2}(L-\ell)\log\left(1+\frac{L-\ell}{4(\mathsf{v}+\lambda)t}\right)\right\}.$$

Proof. Standard generator computations.

Spatial Decoupling

Lemma. (Spatial decoupling)

- Let $(\zeta_t)_{t\geq 0}$ be the CPIP with parameters v and λ.
- Let $\ell \in \mathbb{N}$, $x_1, x_2 \in \mathbb{Z}^d$, $\|x_1 - x_2\| \geq 2\ell + 2$, and $t > 0$.
- Let A_i, $(i = 1, 2)$ be an event whose occurrence depends only on $\{\zeta_s(y) : (y, s) \in B_{x_i}(\ell) \times [0, t]\}$.

Then,

$$|\mathrm{Cov}(\mathbb{1}_{A_1}, \mathbb{1}_{A_2})| \leq 4\mathrm{discr}^{\mathrm{icp}}_{\mathsf{v},\lambda}(\ell, \lfloor \tfrac{1}{2}\|x_1 - x_2\| \rfloor, t),$$

Temporal decoupling is more delicate.

Temporal Decoupling

Baldasso and Teixeira (2018): decoupling estimates for IP $(d = 1)$:

- Let $\mathcal{Q}_1, \mathcal{Q}_2$ be two well-separated space-time boxes:

$$\mathsf{d} := \operatorname{dist}(\mathcal{Q}_1, \mathcal{Q}_2) \geq 6(\operatorname{per}(\mathcal{Q}_1) + \operatorname{per}(\mathcal{Q}_2)) + C_1,$$

- Let $f_1, f_2 : \{0,1\}^{\mathbb{Z}\times\mathbb{R}} \to [0,1]$ be
  - non-decreasing functions;
  - f_i supported on $\mathcal{Q}_i$, for $i = 1, 2$
- Decoupling with sprinkling: for $p < p' \in [0,1]$

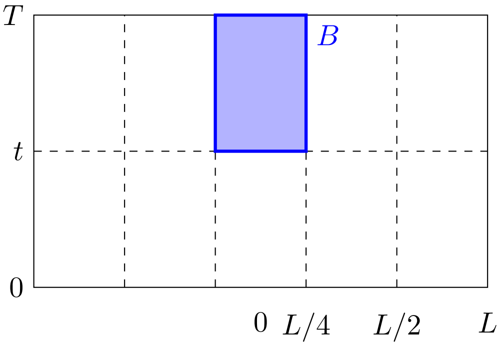
$$\mathbb{E}_{\pi_p}(f_1 f_2) \leq \mathbb{E}_{\pi_{p'}}(f_1)\mathbb{E}_{\pi_{p'}}(f_2) + c_1 \mathsf{d}^2 \exp\left[-c_1^{-1}(p'-p)^2 \mathsf{d}^{1/4}\right],$$

where π_p is the Bernoulli product measure with density p.

Problem. We need a refinement

- valid for all d;
- desintegrated version

Temporal Decoupling

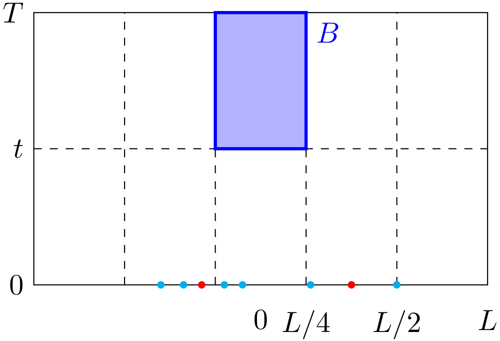


Lemma. (Stochastic domination between IPs)
Given $\xi, \xi' \in \{0,1\}^{\mathbb{Z}^d}$, there exists a coupling such that for any $\ell < L \in \mathbb{N}$, $0 < t \leq T$ and $p \in [0,1]$:

$$\xi'_s(x) \geq \xi_s(x) \quad \text{for all } (x,s) \in B_0(L/4) \times [t,T]$$

outside an event of small probability.

Temporal Decoupling



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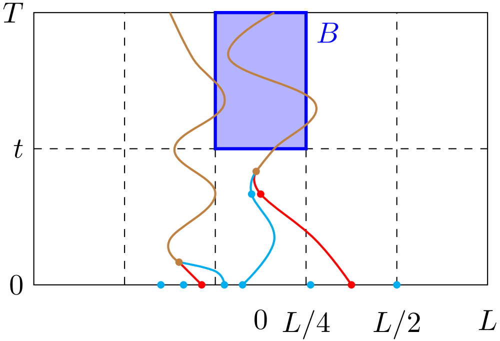

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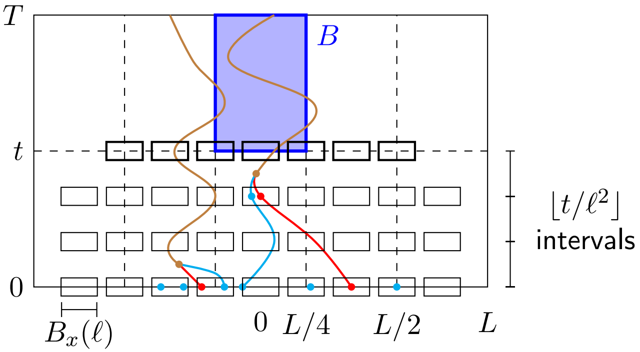

- There are 'less ξ particles than ξ' particles'.
- Every ξ particle that reaches B meets a ξ' particle before t.
- For independent SRWs (rate 1) on $\mathbb{Z}^d$:

$$\mathrm{meet}(\ell) := \inf\{\mathbb{P}_{x,y}(\exists s \le \ell^2 : X_s = X'_s) : \ x, y \in B_0(\ell)\} \ge \frac{c}{\ell^{(d-2)\vee 0}}.$$

Decoupling – error control

- There are 'less ξ particles than ξ' particles':
  Use $p \in (0,1]$ and functions $g^{\uparrow}$ and $g^{\downarrow}$.

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- Every ξ particle that reaches $B(L/4) \times [t,T]$: discrepancy

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- If the above holds:
  - Many attempts for pairing.
  - Union bound: control every particle of $B(L/2)$.

$$\le |B_0(L/2)| \cdot (1 - \mathrm{meet}(\ell))^{\lfloor t/\ell^2 \rfloor}.$$

Temporal Decoupling

Lemma. (Stochastic domination between IPs)
Given $\xi, \xi' \in \{0,1\}^{\mathbb{Z}^d}$, there exists a coupling such that for any $\ell < L \in \mathbb{N}$, $0 < t \leq T$ and $p \in [0,1]$:

$$\xi'_s(x) \geq \xi_s(x) \quad \text{for all } (x,s) \in B_0(L/4) \times [t,T]$$

outside an event of probability at most:

$$g^{\uparrow}(\ell, L, t, p, \xi) + g^{\downarrow}(\ell, L, t, p, \xi') + \mathrm{err}_{\mathrm{coup}}(\ell, L, t, T),$$

where

$$\mathrm{err}_{\mathrm{coup}} := |B_0(L/2)| \cdot (1 - \mathrm{meet}(\ell))^{\lfloor t/\ell^2 \rfloor} + \mathrm{discr}^{\mathrm{ip}}(L/4, L/2, T).$$

Temporal Decoupling

After integrating, we recover Baldasso and Teixeira

Lemma. For $\ell < L \in$, $t > 0$ and $0 \leq p < p' \leq 1$

$$\int_{\{0,1\}^{\mathbb{Z}^d}} g^{\uparrow}(\ell, L, t, p', \xi)\ \pi_p(\mathrm{d}\xi) \quad \text{and} \quad \int_{\{0,1\}^{\mathbb{Z}^d}} g^{\downarrow}(\ell, L, t, p, \xi)\ \pi_{p'}(\mathrm{d}\xi)$$

are both smaller than

$$(2L+1)^d \cdot \Big(e(2\ell+2)^d t + e\Big) \cdot \exp\Big\{-2(2\ell+1)^d (p'-p)^2\Big\}.$$

Renormalization for Extinction ($\mathsf{v} \to \infty$)

Scales and constants.

- $L_N = \alpha^N \cdot L_0$ and $h_N = \alpha^N \cdot h_0 \quad (\alpha = 128)$;
- $L_0 = \sqrt{\mathsf{v}} \log^4(\mathsf{v})$ and $h_0 = 2 \log^3(\mathsf{v})$;
- Fix $p_0 > p$ so that $2dp_0\lambda < 1$, and take, for each $n \geq 1$:

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Scale n. Let $\delta_n := (2C(d,\alpha))^{-n-1}$.
For v large enough (uniformly over n) we establish:

$$(\mathrm{HC}_n) \qquad \begin{array}{c}\xi^{\zeta_0} \text{ is stochastically} \\ \text{dominated by } \pi_{p_n}\end{array} \implies \sup_{x,t} \mathbb{P}_{\lambda,\mathsf{v}}\left(H(\mathcal{Q}_n(x,t))\right) < \delta_n.$$

Lemma. (Horizontal decoupling)
Let $n \geq 1$.
Assume (HC_n) and ξ_0 stochastically dominated by π_{p_n}.
Then $\forall$ $(x, s), (y, t)$ with $\|x - y\| \geq 4L_n$ and $|s - t| \leq 2h_n$:

$$\mathbb{P}\left(H(\mathcal{Q}_n(x, s)) \cap H(\mathcal{Q}_n(y, t))\right) \leq \delta_n^2 + \mathsf{v}^{-2^n}.$$

Lemma. (Vertical decoupling)
Let $n \geq 1$.
Assume (HC_n) and ξ_0 stochastically dominated by $\pi_{p_{n+1}}$.
Then, $\forall (x, s), (y, t)$ with $|s - t| > 2h_n$:

$$\mathbb{P}\left(H(\mathcal{Q}_n(x, s)) \cap H(\mathcal{Q}_n(y, t))\right) \leq \delta_n^2 + \mathsf{v}^{-2^n}.$$

Renormalization for other results

Extinction for CPIP (v $\to 0$).

- Subcritical site percolation:
  Largest cluster in $B(L)$ has size of order $\log L$.
- With high probability, CP dies on a finite graph of size n in a time of order e^{cn}.
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Survival for CPIP (v $\to \infty$). Much more involved:

- Definition of good boxes;
- Suitable propagation of "good boxes" along various scales;
- Control of scale 0 already more delicate.

Remark. Replacing the Interchange Process with the Exclusion Process when v $\to \infty$ seems much harder.

Thank You!