Invariant measures for substitutions on countable alphabets

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Let A be a **countable** set (*alphabet*). Here $A = \mathbb{Z}_+$ (and sometimes $A = \mathbb{Z}$).

A^* be the set of finite words on A

$A^{\mathbb{Z}_+}$ be the set of infinite words on A, where $\mathbb{Z}_+ = \{0, 1, 2, ...\}$

A **substitution** is a map $\sigma : A \to A^*$.

We assume that for every letter $a \in A$, $\sigma(a)$ is not empty.

We extend σ to A^* and $A^{\mathbb{Z}_+}$ by concatenation and, to simplify the notation, we also denote these extensions by σ.

Hence $\sigma(u_0 \ldots u_n) = \sigma(u_0) \ldots \sigma(u_n)$ for all $u_0 \ldots u_n \in A^*$ and $\sigma(u_0 u_1 \ldots) = \sigma(u_0)\sigma(u_1) \ldots$ for all $u_0 u_1 \ldots \in A^{\mathbb{Z}_+}$.

We assume that there exists a letter a in A such that the length of the finite word $\sigma^n(a)$ converges to infinity as n goes to infinity.

To σ, we associate a **shift dynamical system** (Ω_σ, S), where

$$\Omega_\sigma = \{u \in A^{\mathbb{Z}_+} : \text{any finite factor of } u \text{ occurs in } \sigma^n(a) \text{ for some } n \in \mathbb{N} \text{ and } a \in A\}.$$

and S is the **shift map**

$$S(u_0 u_1 \ldots) = u_1 u_2 \ldots \text{ for all } u = u_0 u_1 \ldots \in A^{\mathbb{Z}_+}.$$

Shift dynamical systems of substitutions provide many important examples in ergodic theory and they have been **well studied** mainly in the literature **when the alphabet is finite.**

If σ is a **primitive substitution** on

$$A = \{0, \ldots, d-1\},\ d \geq 2,$$

i.e there exists $k \in \mathbb{N}$ such that for all $a, b \in A$ the letter b occurs in the word $\sigma^k(a)$, then **the dynamical system is minimal and uniquely ergodic.**

Moreover Ω_σ is the closure of the orbit of any periodic point of σ.

Let μ be the **unique shift invariant probability measure**.

On cylinders

$$[w] = \{u_0 u_1 \ldots \in \Omega_\sigma,\ u_i = w_i,\ i = 0, \ldots, n\},$$

for $w = w_0 \ldots w_n$, $w_i \in A$ for $i = 0, ..., n$,

$\mu[W]$ **is the frequency of occurrences of** w **in the periodic point** u**.**

The vector $(\mu[0], \ldots, \mu[d-1])$ is the normalized left Perron eigenvector associated to the dominant Perron-Frobenius eigenvalue of the matrix $M_\sigma = (M_{ij})_{0 \le i,j \le d-1}$ defined as

$M_{ij} := |\sigma(i)|_j$ **is the number of occurrences of the letter** j **in the word** $\sigma(i)$**.**

When A is countably infinite, the situation is more complicated.

We can also define the **countable matrix**

$$M_\sigma = (M_{ij})_{i,j \in \mathbb{Z}_+}$$

by

$$M_{ij} = |\sigma(i)|_j.$$

One of the difficulties in studying ergodic properties of the dynamical system (Ω_σ, S) in such case lies in the fact that the countably infinite matrix M_σ may present a larger number of possible behaviors.

For all $i, j \in A$ and for all integer $n \in \mathbb{N}$,

$$|\sigma^n(i)|_j = M^n_{ij}, \quad |\sigma^n(i)| = \sum_{j=1}^{\infty} M^n_{ij}.$$

For example, if $\sigma(n) = 0(n+1)$ for all $n \in \mathbb{Z}_+$, then

$$M_\sigma = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

$\sigma : A \to A^*$ is of **constant length** (resp. **bounded length**) if there exists an integer $L \geq 1$ such that $|\sigma(a)| = L$ (resp. $|\sigma(a)| \leq L$) for all $a \in A$.

If σ has constant length L (resp. bounded length by L), then the sum of the entries of each line of M^n_σ, $n \in \mathbb{N}$ equals L^n (resp. $\leq L^n$).

Here **we assume that σ is a bounded length substitution**.

Consider a countable non-negative matrix $M = (M_{ij})_{i,j \in \mathbb{Z}_+}$.

Irreducible and aperiodic means $\forall\, i, j \in \mathbb{Z}_+$, $\exists\ n = n(i,j) \geq 1$ such that $M_{ij}^k > 0\ \forall k > n$.

For all $i, j \in \mathbb{Z}_+$, there exists

$$\lim_{n\to\infty} (M_{ij}^n)^{1/n} = \lambda.$$

λ is called the **Perron Value** of M

We say that M is **transient** if and only if $\sum_{n=0}^{+\infty} \frac{M_{ij}^n}{\lambda^n} < +\infty$, otherwise M is said to be **recurrent**.

It is known that if M is recurrent there are left and right eigenvectors l and r associated to λ and when the scalar product $l \cdot r$ is finite, we say that M is **positive recurrent**, otherwise M is said to be **null recurrent**.

For substitutions on countably infinite alphabets an important study was initiated by Ferenczi '06.

For instance, among several results, it is considered the squared **drunken substitution** defined as

$$\sigma(n) = (n-2)nn(n+2), \; n \in 2\mathbb{Z}$$

and proved that the dynamical system (Ω_σ, S) **is not minimal and has no finite invariant measure**.

However it is also shown that (Ω_σ, S) has an infinite invariant measure μ which is shift ergodic.

In Ferenczi '06 it is also proved that if σ **is of constant length, left determined and has an irreducible aperiodic positive recurrent matrix** M_σ, then **the associated shift dynamical system admits an ergodic probability invariant measure.**

σ is **left determined** if there exists a nonnegative integer N such that every w of length at least N which occurs on some element of Ω_σ, has a unique decomposition $w = w_1 \dots w_s$, where each $w_i = \sigma(a_i)$ for some $a_i \in A$, except that w_1 may be only a suffix of $\sigma(a_1)$ and w_s may be only a prefix of $\sigma(a_s)$, and the a_i, $1 \le i \le s-1$, are unique.

Bezugly, Jorgensen and Sanadhya '22 proved that for **a left determined substitution $\sigma : \mathbb{Z} \to \mathbb{Z}$ with M_σ irreducible, aperiodic and recurrent which is also of *bounded size*** (the letters of all $\sigma(n)$ belong to the set $\{n-t, n-t+1, \dots, n+t\}$ where $t \in \mathbb{Z}$ is independent of n),**there exists a shift invariant measure μ on Ω_σ.**

It is also worth mentioning that an arithmetic study of substitutions on countably infinite alphabets was done in Mauduit '06.

Extending Ferenczi '06, we obtain.

Theorem 1

Let $\sigma : \mathbb{Z}_+ \to \mathbb{Z}_+^*$ *be a* **bounded length substitution** *such that* σ *has a periodic point* u *and* $M = M_\sigma$ **is irreducible and aperiodic**. *If* M *satisfies*

$$\lim_{n \to +\infty} \sup_{i \in \mathbb{Z}_+} \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} = 0 \text{ for all } j \in \mathbb{Z}_+, \tag{1}$$

then the dynamical system (Ω_σ, S) **has no finite invariant measure.**

It is important to notice that condition (1) **may or may not hold on both the transient and the null recurrent cases.**

One natural question is if condition (1) can be replaced by the weaker condition

$$\lim_{n\to+\infty} \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} = 0 \text{ for all } i, j \in \mathbb{Z}_+. \tag{2}$$

The last condition is more natural and holds for a large class of substitutions σ such that M_σ is transient or null recurrent and σ has constant length, or M_σ is positive recurrent with left Perron eigenvector $l = (l_k)_{k\geq 0} \notin l^1$.

Let us consider some examples where M is a multiple of an irreducible stochastic matrix P.

(1) is equivalent to

$$\lim_{n\to+\infty} \sup_{i\in A} P_{ij}^n = 0 \text{ for all } j \in A.$$

Consider $A = \mathbb{Z}$ and set

$$P_{-n,-n-1} = q_n = 1 - P_{-n,-n+1},\ \ n \geq 1,$$

where $q_n \in (0,1)$ and $\sum_{n=1}^{+\infty} q_n < \infty$.

Also put $P_{0,-1} = P_{0,1} = 1/2$ and $P_{m,-n} = 0$ for $m, n \geq 1$. No matter how we complete the definition of P to obtain a irreducible and aperiodic matrix which may be recurrent or transient, we have that

$$\sup_{a\in A} P^n_{a,0} \geq P^n_{-n,0} \geq \prod_{k=1}^{+\infty}(1-q_k) > 0, \quad \text{for every } n \geq 1.$$

Thus (1) does not hold.

However, since $\lim_{n\to\infty} q_n = 0$, there is no multiple of P which is a matrix M associated to a substitution.

We will provide another example.

Again $A = \mathbb{Z}$. Set

$$P_{-2^n,0} = 1/2 = P_{-2^n,-2^n-1} \text{ and } P_{-2^n-j,-2^n-j-1} = 1$$

for $j = 1, ..., 2^n - 1$ and $n \geq 1$.

We can check that $P^{2^n}_{-2^n-1,0} = 1/2$. Again, no matter how we complete the definition of P, which may be recurrent or transient, we have that

$$\limsup_{n\to\infty} \sup_{a\in A} P^n_{a,0} \geq 1/2 > 0,$$

Define

$$M_{-2^n,0} = 1 = M_{-2^n,-2^n-1} \text{ and } M_{-2^n-j,-2^n-j-1} = 2$$

and complete the definition for the other entries to obtain an irreducible and aperiodic matrix associated to a substitution of constant length two. We have $P = M/2$ and (1) does not hold.

As a third example, take P as the transition matrix of a SRW on $\mathbb{Z}$.

Fix $p \in (0,1)$ and set

$$P_{n,n+1} = p = 1 - P_{n,n-1} \;\; n \in \mathbb{Z}.$$

The SRW is irreducible with period two, null recurrent if $p = 1/2$ and transient otherwise.

We can use P^2 instead of P for an example with an aperiodic chain.

For $p = 1/2$, a standard computation using the binomial distribution and Stirling formula shows that

$$\sup_{w \in \mathbb{Z}} P^n_{w,\tilde{w}} = \sup_{w \in \mathbb{Z}} P^n_{0,\tilde{w}-w} \leq \max_{0 \leq k \leq n} \binom{n}{k} p^k (1-p)^{n-k} = O(n^{-\frac{1}{2}}).$$

Here, we also have $P = M/2$, where M is a substitution matrix of constant length equals to 2 defined as

$$M_{n,n+1} = M_{n,n-1} = 1 \; \forall \, n \in \mathbb{Z}.$$

Thus M satisfies (1).

Lemma 2

Let $M = (M_{ij})_{i,j\in\mathbb{Z}_+}$ be a nonnegative, irreducible and aperiodic matrix with finite Perron value λ.

If M **is transient with constant line sums, or** *M* **is positive recurrent with a left Perron eigenvector** *$l = (l_k)_{k\geq 0} \notin l^1$, then*

$$\lim_{n\to+\infty} \frac{M^n_{ij}}{\sum_{k=0}^{+\infty} M^n_{ik}} = 0 \text{ for all } i,j \in \mathbb{Z}_+.$$

Let $\sigma := \sigma_{a,b,c}$ defined by

$$\sigma(0) = 0^{a+b}1^c \text{ and } \sigma(n) = (n-1)^a n^b (n+1)^c \text{ for all } n \geq 1$$

where a, b, c are nonnegative integers such that $a > 0$, $c > 0$ and $i^k = ii \ldots i$ (k times).

The matrix

$$M_\sigma = \begin{bmatrix} a+b & c & 0 & 0 & 0 & 0 & 0 & \cdots \\ a & b & c & 0 & 0 & 0 & 0 & \cdots \\ 0 & a & b & c & 0 & 0 & 0 & \cdots \\ 0 & 0 & a & b & c & 0 & 0 & \cdots \\ 0 & 0 & 0 & a & b & c & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

is irreducible and aperiodic.

σ is a substitution of constant length $L = a + b + c$.

The stochastic matrix $P = M_\sigma / L$ is the transition matrix of a homogeneous nearest-neighbor random walk in $\{0, 1, 2, ...\}$ partially reflected at the boundary.

This random walk is (in the probabilistic sense) positive recurrent if $c < a$, null recurrent if $c = a$ and transient if $c > a$.

The difference for the matrix theoretical definition is that we also have null recurrence in the case $c > a$.

Proposition 1

The following properties hold:

- *If $c < a$, then M_σ is positive recurrent.*
- *If $c \geq a$, then M_σ is null recurrent and (Ω_σ, S) has no finite invariant measure.*

Consider the case $c > a$.

We show that $\lambda_P < 1$ and that $\overline{P}_{00}(1/\lambda_P) = \infty$, this implies null recurrence.

Furthermore M_σ satisfies (1) for every $a \leq c$ and b.

Thus (Ω_σ, S) has no finite invariant measure.

In the case $a = c$ we can also show that M_σ satisfies (1).

Remark: Small modifications on the matrix can completely change its behavior. For instance, consider the case $b = 0$ and $a = c$ which implies that M_σ is null recurrent. Instead of $\sigma(0) = 0^a 1^c$, put $\sigma(0) = 1^c$, then we have that M_σ is transient. For the case $b > 0$, $a \leq c$ and $\sigma(0) = 1^c$, we also have transience.

Remark: We consider the substitution σ of ferenczi '06 defined on $A = \mathbb{Z}$ by $\sigma(n) = (n-1)nn(n+1)$. The associated matrix is null recurrent and satisfies condition (1). Hence by using Theorem 1, we deduce that the dynamical system (Ω_σ, S) associated to σ has no finite invariant measure.

Let $\sigma := \sigma_{a_n,b_n,c_n}$ defined by

$$\sigma(0) = 0^{a_0+b_0}1^{c_0} \text{ and } \sigma(n) = (n-1)^{a_n} n^{b_n} (n+1)^{c_n} \text{ for all } n \geq 1$$

where a_n, b_n, c_n are nonnegative integers such that $a_n > 0$, $c_n > 0$ for every $n \geq 1$ and $L = \sup\{a_n + b_n + c_n : \ n \geq 1\} < \infty$. The matrix M_σ is irreducible and aperiodic with bounded length L and can be represented as

$$M_\sigma = \begin{bmatrix} a_0+b_0 & c_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_1 & b_1 & c_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_2 & b_2 & c_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & a_3 & b_3 & c_3 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_4 & b_4 & c_4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We show that (Ω_σ, S) is not minimal for these substitutions.

Theorem 3

Let σ be a **bounded length substitution** *on $A = \mathbb{Z}_+$ such that*

M_σ is irreducible, aperiodic, **positive recurrent**,

then the dynamical system (Ω_σ, S) **has a shift invariant measure μ which is finite if and only if any left Perron eigenvector l belongs to l^1.**

Remark: Theorem 3 improves a result of Bezugly, Jorgensen and Sanadhya '22, where it is assumed the additional hypothesis that σ is a bounded size left determined substitution.

Let $P = (P_{ij})_{i,j \geq 0}$ be a **stochastic matrix.**

P is

- **ergodic** (positive recurrent) if

$$\lim_{n \to \infty} P_{ij}^n = \pi_j > 0 \quad \forall\, i, j \in \mathbb{N},$$

  where $(\pi_j)_{j \geq 0}$ is a probability vector;

- **strongly ergodic** if P is ergodic and if

$$\lim_{n \to \infty} \sup_{i \geq 0} \sum_{j=0}^{\infty} |P_{ij}^n - \pi_j| = 0.$$

A nonnegative matrix $M = (M_{ij})_{i,j \geq 0}$ is said to be **scrambling** if there exists $a > 0$ such that

$$\sum_{j=0}^{+\infty} \min(M_{ij}, M_{kj}) \geq a \text{ for all } i \neq k \in \mathbb{Z}_+.$$

$P = (P_{ij})_{i,j \geq 0}$ is **strongly ergodic if and only if a power of P is scrambling.**

Let $M = (M_{ij})_{i,j \geq 0}$ be nonnegative, **irreducible, aperiodic and positive recurrent with finite Perron value $\lambda > 0$.**

Let $P = (P_{ij})_{i,j \geq 0}$ be the **stochastic matrix** defined by

$$P_{ij} = \frac{M_{ij} r_j}{\lambda r_i} \text{ for all } i, j \geq 0,$$

where $r = (r_k)_{k \geq 0}$ **is a right Perron eigenvector of** M**.**

We say that M **is strongly ergodic if** $P = (P_{ij})_{i,j \geq 0}$ **is too**.

M is **strongly ergodic** if and only if

(1) there exists positive integer n and a vector of probability $(\pi_j)_{j\geq 0}$ such that

$$\lim_{n\to\infty}\sup_{i\geq 0}\sum_{j=0}^{+\infty}\left|\frac{M_{ij}^n r_j}{\lambda^n r_i}-\pi_j\right|=0, \tag{3}$$

furthermore $\pi_j = l_j r_j$ where l and r are respectively Perron left and right eigenvectors such that $l\cdot r = 1$.

(2) there exists a positive integer n and $a>0$ such that $\forall\, i\neq k$, we have

$$\sum_{j=0}^{+\infty} min\Big(\frac{M_{ij}^n r_j}{\lambda^n r_i},\frac{M_{kj}^n r_j}{\lambda^n r_k}\Big)\geq a. \tag{4}$$

Assume that M has a right Perron eigenvector $r=(r_i)_{i\geq 0}\in l^\infty$ which satisfies $\inf\{r_i : i\geq 0\}>0$, then M is strongly ergodic if and only if there exists a positive integer n such that M^n is scrambling.

Theorem 4

Let σ be a **constant length substitution** *on $A = \mathbb{Z}_+$ such that σ has a periodic point u and M_σ is irreducible and aperiodic.*

If there exists a positive integer n such that M_σ^n **is scrambling**, *then $\exists$* **a unique probability shift invariant measure of** (Ω_σ, S).

Theorem 5

Let σ be a **non-constant bounded length substitution on** $A = \mathbb{Z}_+$ *with a periodic point u and such that*

$M = M_\sigma$ *is* **irreducible, aperiodic, positive recurrent**

with right Perron eigenvector $r = (r_i)_{i \geq 0} \in l^\infty$ *and* $\exists$ n **such that** M_σ^n **is scrambling**,

then the dynamical system (Ω_σ, S) **has a unique invariant probability measure.**

Proposition 2

Let σ be a **bounded length substitution** *on $A = \mathbb{Z}_+$ such that σ has a periodic point u and M_σ is irreducible and aperiodic.*

Assume that there exists an integer n such that M_σ^n **is scrambling**,

then (Ω_σ, S) is minimal.

Examples:

(1) Let σ (infinite Fibonacci) given by

$$\sigma(2n) = 0(2n+1),\ \sigma(2n+1) = 2n+2 \text{ for all } n \geq 0.$$

We can prove by induction that

$$|\sigma^n(0)| = F_n \text{ and } |\sigma^n(0)|_0 = F_{n-2} \text{ for all } n \geq 1,$$

where $(F_n)_{n\geq 0}$ is the Fibonacci sequence defined by

$$F_0 = 1, F_1 = 2,\ F_n = F_{n-1} + F_{n-2} \ \text{ for all } n \geq 2.$$

The substitution matrix is given by

$$M_\sigma = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is irreducible, aperiodic and its Perron eigenvector is the Golden number

$$\beta = \frac{1+\sqrt{5}}{2} = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}.$$

A Perron right and a left eigenvectors are respectively

$$l = (1, 1/\beta, \ldots, 1/\beta^n, \ldots) \text{ and } r = (1, 1/\beta, 1, 1/\beta, 1, 1/\beta, \ldots).$$

Hence M_σ is positive recurrent.

Furthermore M_σ^2 is scrambling, $r \in l^\infty$ and σ has a fixed point $u = \lim_{n\to\infty} \sigma^n(0)$, thus the dynamical system (Ω_σ, S) has a unique probability invariant measure.

(2) Let τ be given by

$$\tau(n) = 0^{a_n}(n+1), \text{ for all } n \geq 0.$$

where $0 \leq a_i \leq C$ for all $i \geq 0$ for some fixed $C > 0$ and $a_0 > 0$ and

$$\lim sup\ a_n \geq 1.$$

The substitution matrix is given by

$$M_\tau = \begin{pmatrix} a_0 & 1 & 0 & 0 & 0 & 0 & \dots \\ a_1 & 0 & 1 & 0 & 0 & 0 & \dots \\ a_2 & 0 & 0 & 1 & 0 & 0 & \dots \\ a_3 & 0 & 0 & 0 & 1 & 0 & \dots \\ a_4 & 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The Perron eigenvalue of M_τ is the unique real number $\lambda > 1$ satisfying

$$1 = \sum_{i=0}^{\infty} a_i \lambda^{-i-1}.$$

A right Perron and a left Perron eigenvector are respectively

$$l = (1, 1/\lambda, \ldots, 1/\lambda^n, \ldots) \text{ and } r = (1, \alpha_1, \ldots, \alpha_n, \ldots),$$

where

$$\alpha_n = \lambda^n - \sum_{i=0}^{n-1} a_i \lambda^{n-i-1} = \sum_{i=1}^{+\infty} a_{n+i-1} \lambda^{-i} \text{ for all } n \geq 1.$$

$l \cdot r$ is finite and M_σ is positive recurrent.

If there exists $k \geq 1$ such that $a_{kn} \geq 1$ for all $n \in \mathbb{Z}_+$, then $\inf\{\alpha_n,\ n \in \mathbb{Z}_+\} > 0$ and M_τ^k is scrambling.

Furthermore τ has a fixed point $u = \lim_{n\to\infty} \tau^n(0)$, thus the dynamical system (Ω_u, S) has a unique probability invariant measure.

Let $M = (M_{ij})_{i,j\geq 0}$ be a nonnegative, irreducible, aperiodic, **positive recurrent and bounded length**.

We say that M is $\star$-**ergodic** if $\forall i, j \in \mathbb{Z}_+$,

$$\lim_{n\to+\infty} \frac{M^n_{ij}}{\sum_{k=0}^{+\infty} M^n_{ik}} = z_j > 0, \tag{5}$$

where the vector $(z_j)_{j\geq 0}$ has 1 as coordinates sum.

We say that M is $\star$ **strongly ergodic** if there exists a vector $(z_j)_{j\geq 0}$ of positive real numbers such that $\sum_{j=0}^{+\infty} z_j = 1$ and

$$\lim_{n\to\infty} sup_{i\geq 0} \sum_{j=0}^{\infty} \left| \frac{M^n_{ij}}{\sum_{k=0}^{+\infty} M^n_{ik}} - z_j \right| = 0.$$

Question: Is $M = (M_{ij})_{i,j \geq 0}$ $\star$-ergodic equivalent to M positive recurrent with left Perron eigenvector in l^1?

The last question has a positive answer when M is a multiple of a stochastic matrix.

Question: Does there exist a nonnegative matrix $M = (M_{ij})_{i,j \geq 0}$ which is strongly ergodic (resp. $\star$-strongly ergodic), but not $\star$-strongly ergodic (resp. strongly ergodic)?

Proposition 3

Let $M = (M_{ij})_{i,j \geq 0}$ be an irreducible, aperiodic matrix with finite Perron value λ.

and M has a **right Perron eigenvector** $r = (r_i)_{i \geq 0} \in l^\infty$ **satisfying** $\inf\{r_i,\ i \in \mathbb{Z}_+\} > 0$**.**

If M **is strongly ergodic, then** M **is** $\star$ **strongly ergodic.**

Lemma 6

Let $M = (M_{ij})_{i,j\geq 0}$ be a $\star$-ergodic matrix with finite Perron value λ and with right Perron eigenvector $r = (r_i)_{i\geq 0}$, then any left Perron eigenvector of M belongs to l^1, moreover

$$\lim_{n\to\infty} \sum_{k=0}^{+\infty} \frac{M_{ik}^n}{\lambda^n} = \sum_{k=0}^{+\infty} \lim_{n\to\infty} \frac{M_{ik}^n}{\lambda^n} = c\, r_i, \quad \forall i \in \mathbb{Z}_+, \tag{6}$$

for some $c > 0$, and

$$\lim_{n\to+\infty} \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} = \frac{l_j}{\sum_{k=0}^{+\infty} l_k} > 0, \quad \forall i,j \in \mathbb{Z}_+. \tag{7}$$

Theorem 7

Let σ *be a* **nonconstant bounded length substitution** *on* $A = \mathbb{Z}_+$

such that σ *has a periodic point* u *and* M_σ *is irreducible, aperiodic.*

If M_σ **and** $M_{\sigma_t},\ t \geq 2$ **are** $\star$**-strongly ergodic,**

then the dynamical system (Ω_σ, S) **has a unique probability shift invariant measure.**

Thank you