Applications of a Sprinkled Decoupling Inequality in Hammersley's Particle System

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28-04-2025

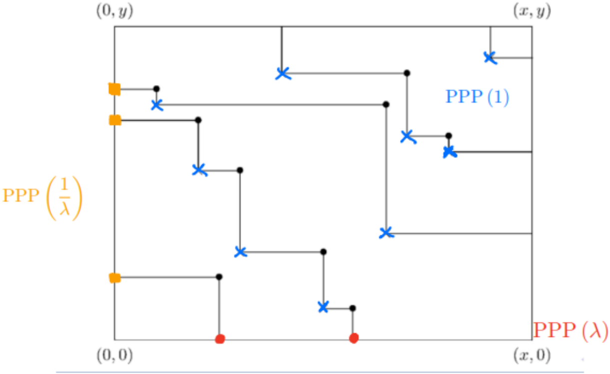
This is a joint work with

Leandro P. R. Pimentel - UFRJ

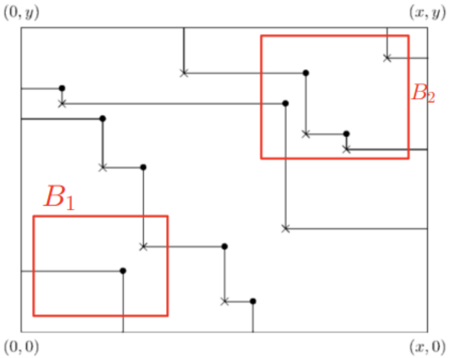
Outline

Main Features of HPS

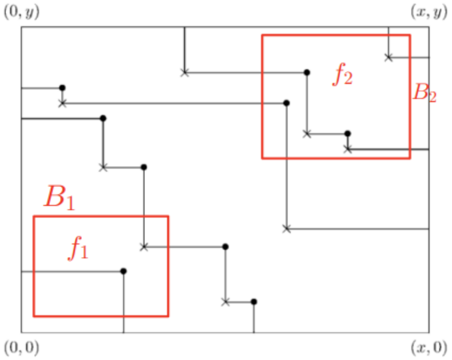
- Initial configuration: Poisson point process (PPP) of intensity λ on $\mathbb{R}$.
- Dynamics: whenever a Poisson mark appears, the closest particle to its right jumps to its location.
- Stationarity: at any fixed time, the configuration of particles is again a PPP of intensity λ.



Theorem 1 (Sprinkled Decoupling Inequality)



Theorem 1 (L. Pimentel, V.)



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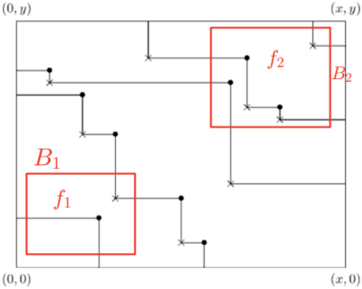


f_1, f_2 monotone functions on B_1 and B_2.
$d = \mathrm{d}(B_1, B_2) > C\epsilon^{-1}(\mathrm{per}(B_1) + \mathrm{per}(B_2))$,

- If f_1, f_2 non-increasing, then $\forall \rho > \rho'$ s.t $\rho - \rho' = \epsilon$
  $\mathbb{E}^{\rho}[f_1 f_2] \leq \mathbb{E}^{\rho'}[f_1]\mathbb{E}^{\rho'}[f_2] + 10e^{-c_I \epsilon^4 d}$
- If f_1, f_2 non-decreasing, then $\forall \rho' > \rho$ s.t $\rho' - \rho = \epsilon$
  $\mathbb{E}^{\rho}[f_1 f_2] \leq \mathbb{E}^{\rho'}[f_1]\mathbb{E}^{\rho'}[f_2] + 10e^{-c_I \epsilon^4 d}$

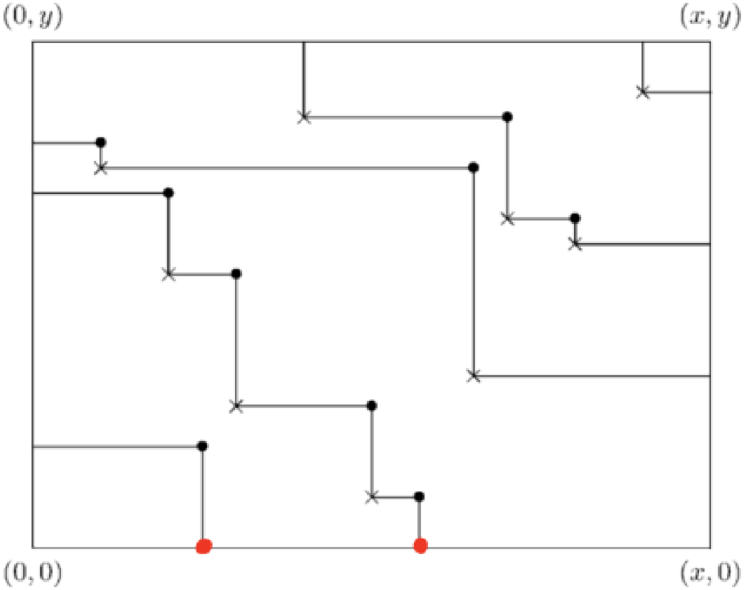
Detection Problem

- Scenario: a target moves in the dynamic environment formed by HPS detectors.
- Target also jumps only at integer times and occupies integer positions within a fixed range.
- Objective: can the target avoid detection over time?

Result:

- A target starting at the origin, can avoid detectors indefinitely, with positive probability.

Detection Problem



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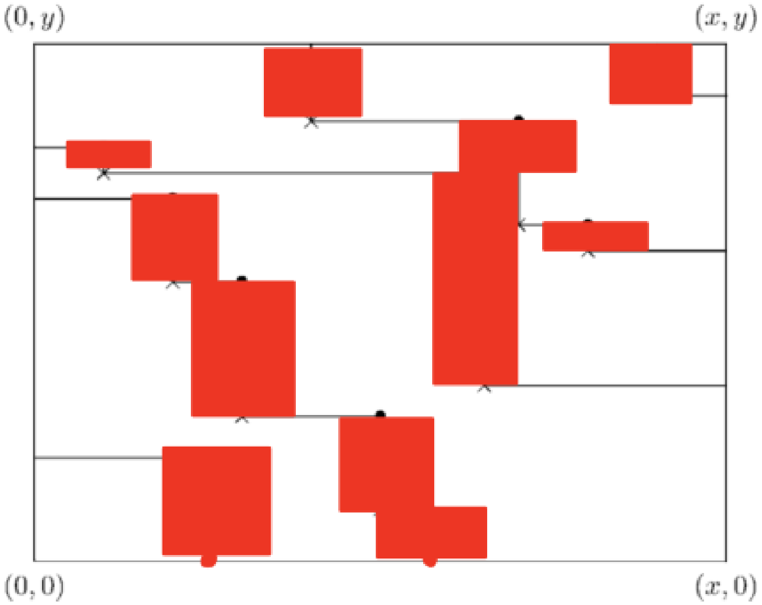

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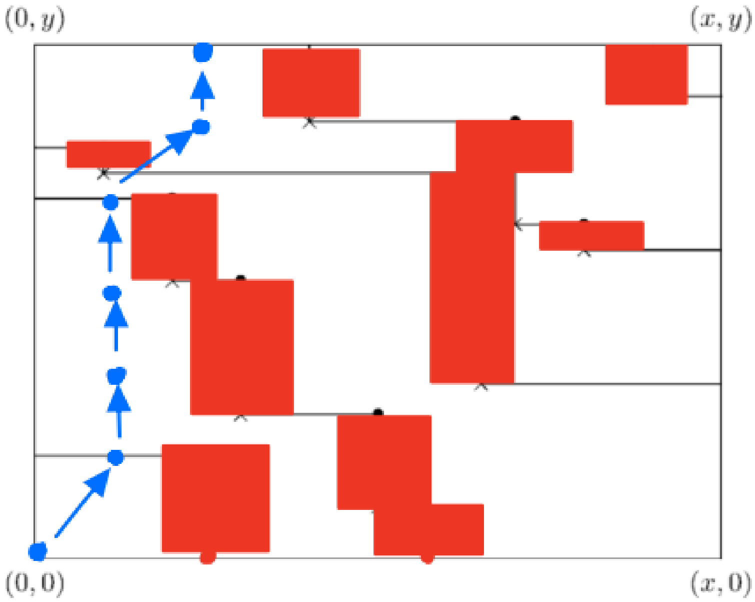
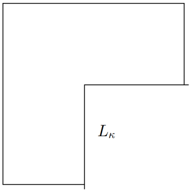


Theorem 2 (L. Pimentel, V.)



$$l_0 = 10^{100} \ , \ l_{k+1} = \lfloor l_k^{1/2} \rfloor l_k \ \text{ and } \ L_k = \left\lfloor \left(\frac{3}{2} + \frac{1}{k} \right) l_k \right\rfloor \ .$$

$$D_k := \{\text{there exists no open crossing } A_k\} \ .$$

$$p_k = \mathbb{P}^{\rho_k} [D_k] \ .$$

$$\rho_{k+1} - \rho_k = c_\rho l_k^{-\delta}$$

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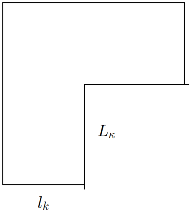

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If

$$p_k \leq l_k^{-4} \text{ for some } k \geq \tilde{k} \,,$$

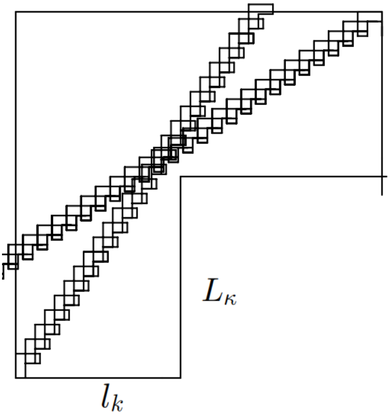
then

$$p_n \leq l_n^{-4} \text{ for all } n \geq k \,.$$

In particular,

$$\mathbb{P}^{\rho}\left[\text{there exists an infinite open path}\right] = 1 \,.$$

Sketch of the proof (based on [1])



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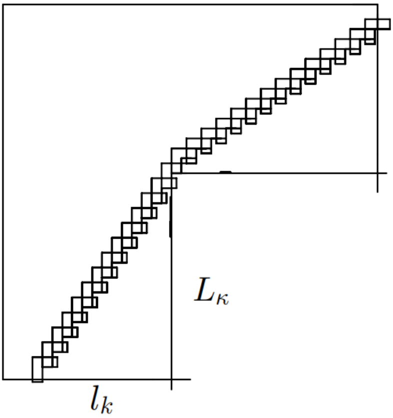

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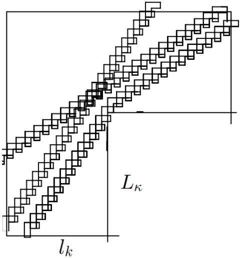

1. $\# \ulcorner = 10 l_{k-1}^{1/2}$;
2. If D_k happens, there exists two $\ulcorner$ and $\ulcorner'$ (one in each group) such that $D_{k-1}(\ulcorner)$ and $D_{k-1}(\ulcorner')$ happen and

$$d\left(\ulcorner, \ulcorner'\right) \geq \left(per(\ulcorner) + per(\ulcorner')\right)^{3/2-\upsilon}$$

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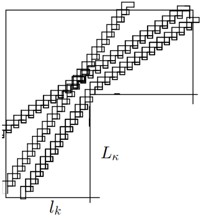

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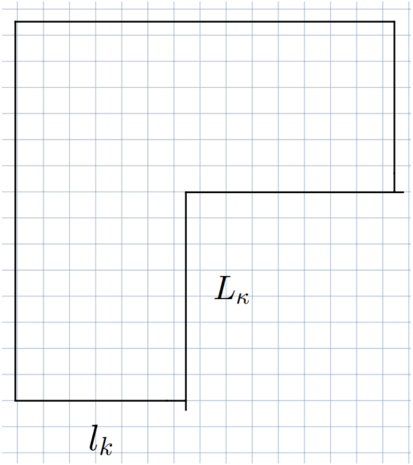
$$\Longrightarrow \; p_k \leq (\#\ulcorner)^2 \left(p_{k-1}^2 + 10e^{-c_l \epsilon^4 d}\right)$$

$$\Longrightarrow \; p_{n+1} l_{n+1}^4 < 1.$$

Trigger inequality



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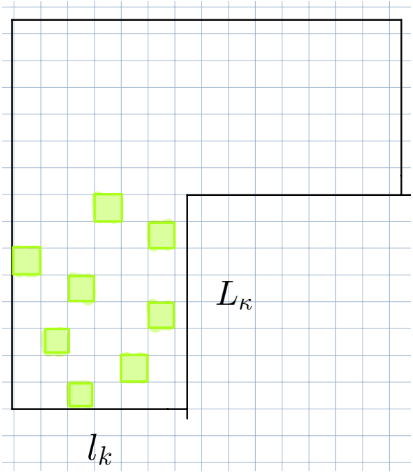

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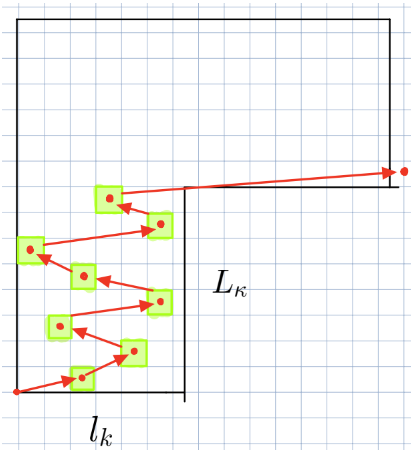

Trigger inequality

$$J_{(x,n)} = \{\text{target is not allowed to occupy } (x, n)\}$$

$$\mathbb{P}^{\lambda_k}\left[\tilde{J}_{(x,n)}\right] = 1 - e^{-2r\lambda_k}e^{-2r} = 1 - e^{-2r(\lambda_k+1)} \leq 1 - e^{-2r(\lambda_0+1)},$$

$$\implies p_k \leq (L_k + 1)\left(1 - e^{-2r(\lambda_0+1)}\right)^{\lfloor l_k/2r \rfloor}$$

Random Walk in HPS

Define the random walk $(X_n)_{n\geq 0}$ on $\mathbb{Z}$ starting at $X_0 = 0$ and evolving on the top of η as follows:

- If $\eta_n(I_{X_n}) > 0$ then
$$\begin{cases} X_{n+1} = X_n + 1 \text{ with probability } p_\bullet \\ X_{n+1} = X_n - 1 \text{ with probability } 1 - p_\bullet \end{cases};$$
- If $\eta_n(I_{X_n}) = 0$ then
$$\begin{cases} X_{n+1} = X_n + 1 \text{ with probability } p_\circ \\ X_{n+1} = X_n - 1 \text{ with probability } 1 - p_\circ \end{cases}.$$

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$I_x = (x - \frac{1}{2}, x + \frac{1}{2})$
$0 \leq p_\circ, p_\bullet \leq 1$

Questions About the Model

- Law of Large Numbers (LLN): Does the walker have a deterministic linear speed?
- Central Limit Theorem (CLT): Does the walker satisfy Gaussian fluctuations around the mean behavior?
- Scaling Limits: How do fluctuations behave at different spatial and temporal scales?

Graphical construction

Consider the coupled family $\{(X_t^w)_{t\geq 0},\ w \in \mathcal{L}\}$ where

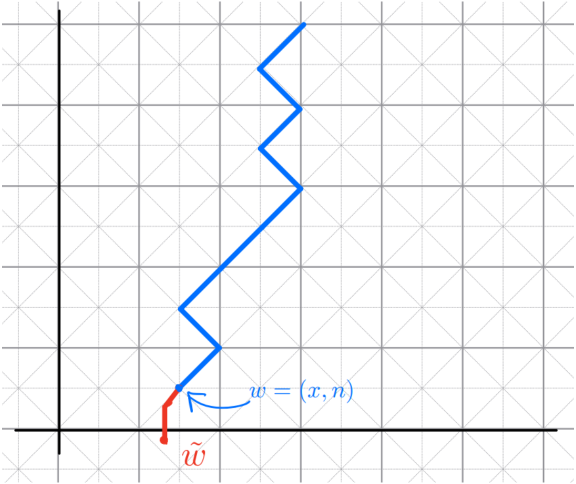
$$\mathcal{L} = \{(x, n) \in \mathbb{Z}^2 : x, n \text{ even or } x, n \text{ odd}\},$$

constructed as $X_0^w = w$ and the position after one step is

$$X_1^w = \begin{cases} x + 2\,\mathbf{1}_{\{U_w \leq p_\bullet\}} - 1, & \text{if } \eta_n^\lambda(I_x) > 0, \\ x + 2\,\mathbf{1}_{\{U_w \leq p_\circ\}} - 1, & \text{if } \eta_n^\lambda(I_x) = 0, \end{cases} \tag{3.3}$$

where $\{U_w : w \in \mathcal{L}\}$ is an i.i.d. collection of uniform random variables in $[0, 1]$ and $\eta_n^\lambda(I_x)$ denotes the number of particles in the interval $(x - 1/2, x + 1/2)$ at time n in the environment with density λ.
For any integer $m \geq 1$, X_m^w is defined by induction.

Graphical construction



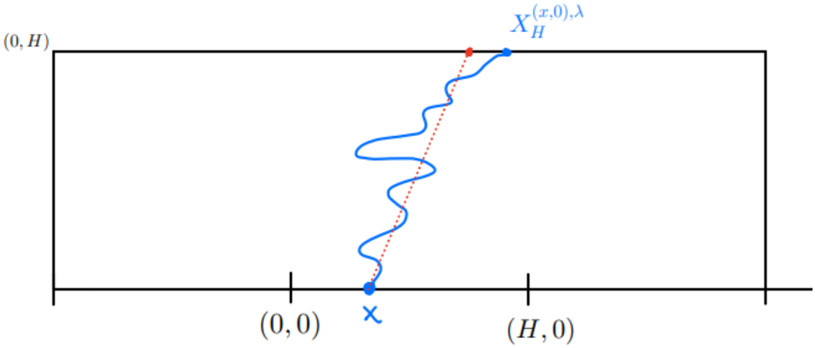
Graphical construction

This family satisfies three fundamental properties:

- **Coalescence:** if two random walks meet at some vertex, they will follow the same trajectory from that point onward;
- **Monotonicity in Space:** If one random walk starts to the right of another, it remains to the right for all future times.
- **Monotonicity in Density:** If two random walks start at the same position and evolve in environments with densities $\lambda < \lambda'$, then, provided $p_\bullet > p_\circ$, the walk in the denser environment (λ') remains to the right of the one in the sparser environment (λ) at all times.

Consider the following event

$$A_H(v,\lambda) := \left\{ \exists\, x \in \ [0,H] \text{ s.t. } X_H^{(x,0),\lambda} - x \geq vH \right\}$$



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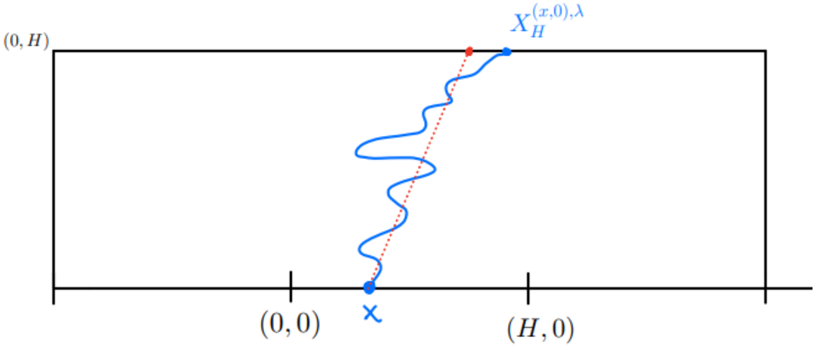

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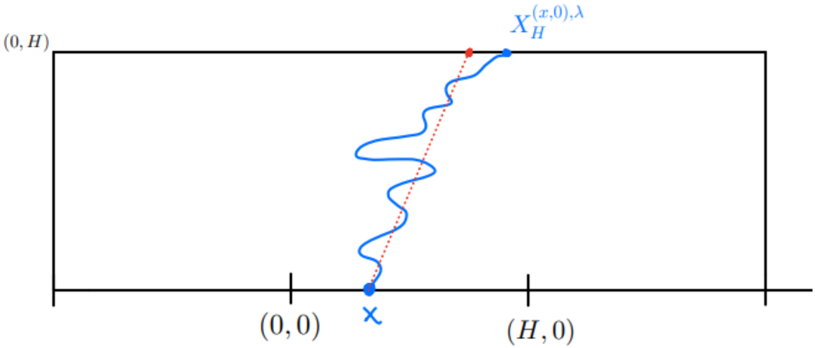
$$p_H(v,\rho) := \mathbf{P}\left(A_H(v,\lambda)\right)$$

$$v_+(\rho) := \inf\left\{ v \in \mathbb{R} \ : \ \liminf_{H\to\infty} p_H(v,\rho) = 0 \right\},$$



Consider the following event

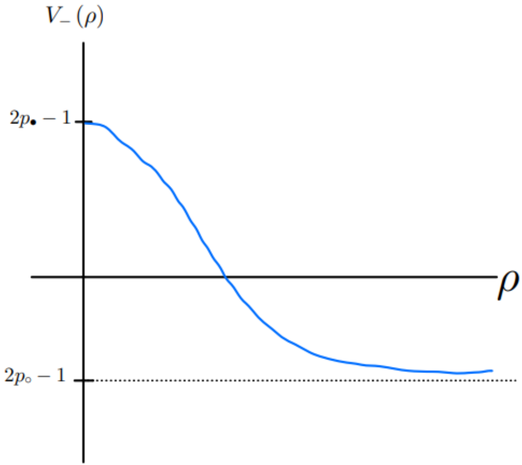
$$\tilde{A}_H(v,\lambda) := \left\{ \exists\, x \in \ [0,H] \text{ s.t. } X_H^{(x,0),\lambda} - x \leq vH \right\}$$

$$\tilde{p}_H(v,\rho) := \mathbf{P}\left(\tilde{A}_H(v,\lambda)\right)$$

$$v_-(\rho) := \sup\left\{ v \in \mathbb{R} \,:\, \liminf_{H\to\infty} \tilde{p}_H(v,\rho) = 0 \right\} ,$$

Properties of $v_\pm(\rho)$ (If $p_\circ < p_\bullet$)

- $\rho \mapsto v_\pm(\rho)$ is non-increasing.
- $\lim_{\rho \to 0} v_\pm(\rho) = 2p_\bullet - 1$.
- $\lim_{\rho \to \infty} v_\pm(\rho) = 2p_\circ - 1$



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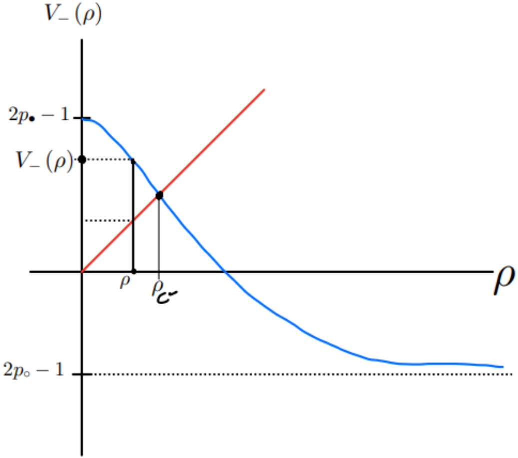


Theorem 3 (L. Pimentel, V.)

$$\rho_{c+} := \inf\{\rho > 0 \,:\, v_+(\rho) < \rho\} \in (0,\infty)\,.$$

$$\rho_{c-} := \inf\{\rho > 0 \,:\, v_-(\rho) < \rho\} \in (0,\infty)\,.$$

Assume that $p_\circ < p_\bullet$. For any $\rho < \rho_{c-}$ we have that $v_-(\rho) > \rho$ and there exist $\tilde{c}_1 \in (0,\infty)$ and $\tilde{v} > \rho$ such that

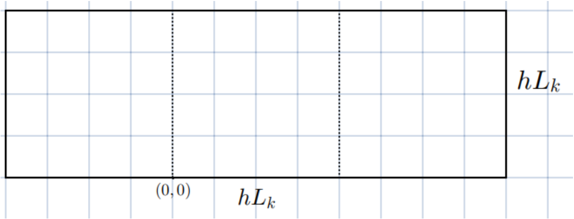
$$\tilde{p}_H(\tilde{v},\rho) \leq \tilde{c}_1 \exp\left(-2\log^{3/2} H\right)\,,\ \forall\, H \geq 1\,.$$

For any $\rho > \rho_{c+}$ we have that $v_+(\rho) < \rho$ and there exist $c_1 \in (0,\infty)$ and $v < \rho$ such that

$$p_H(v,\rho) \leq c_1 \exp\left(-2\log^{3/2} H\right)\,,\ \forall\, H \geq 1\,.$$

Sketch of the Ballisticity Proof (based on [2])

Define a smart sequence of scales L_k, speeds v_k and densities ρ_k...



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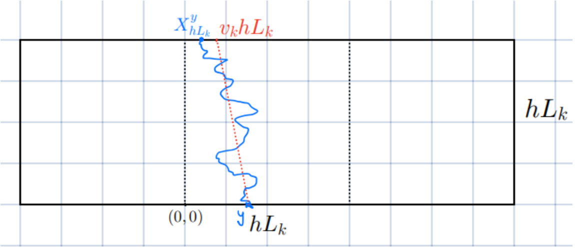

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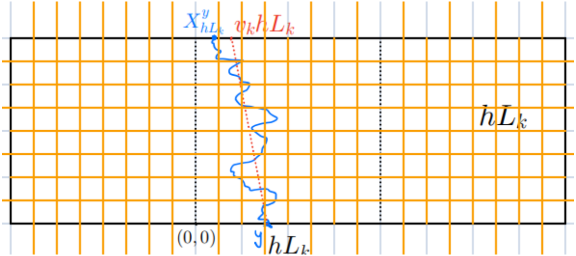

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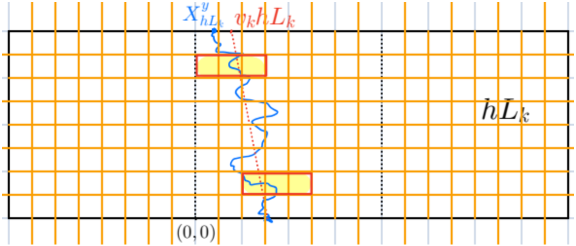

References

[1] R. Baldasso and A. Texeira. How can a clairvoyant particle escape the exclusion process *Ann. Inst. H. Poincaré Probab. Statist.* **54**, 2177–2202 (2018).

[2] M. Hilário, D. Kious. A. Texeira. Random walk on the simple symmetric exclusion process. *Commun. Math. Phys.* **379**, 61–101 (2020).

[3] L. Pimentel, R. Viveros. Sprinkled Decoupling for Hammersley's Particle System (in preparation)