Sprinkled Decoupling for Hammersley's Particle System

Leandro P. R. Pimentel
in collaboration with Roberto Viveros

Instituto de Matemática - UFRJ

Instituto de Matemática
Universidade Federal do Rio de Janeiro

Hammersley's Particle System (HPS)

- Particles are initially distributed as a homogeneous Poisson point process (PPP) on $\mathbb{R}$ with rate $\lambda > 0$.
- A particle at $z \in \mathbb{R}$ jumps to the left at rate given by the distance to its nearest particle to the left, say z^*, and the new location is chosen uniformly between z and z^*.
- $A \subseteq \mathbb{R} \mapsto \eta_t^\lambda(A) =$ "number of particles within A at time t".
- Particles remains distributed as a PPP with the same rate (stationary regime).

Hammersley's Particle System (HPS)

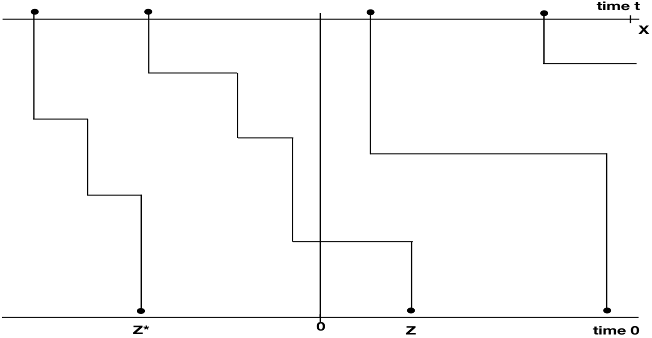

Figure: Particles are represented by bullets (•). Time goes up and trajectories according to Hammersley's process are represented as broken lines attached to each particle.

Motivation

Two models on the top of HPS

- Detection.
- Random Walk in Dynamic Random Environment (RWDRE).
- More details in Roberto's talk (28/4).

Detection

- Suppose particles are detectors that are capable of detecting all targets within some distance $r > 0$ from their location.
- A target can jump up to a certain range $N \geq 1$ only at discrete times, and can predict the future movement of all detectors.
- Question: a target starting at the origin can avoid detection forever with positive probability?

Detection

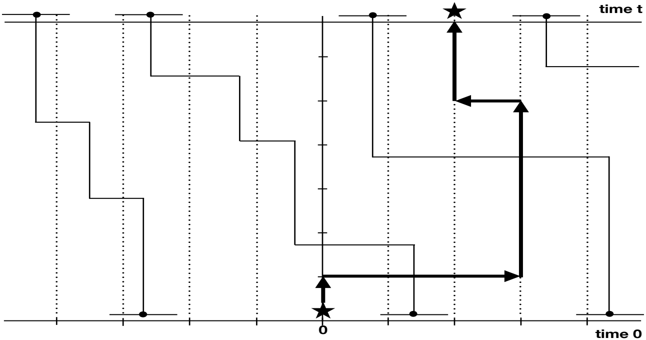

Figure: Line segments centered at • represent detectors. A target starting at the origin is represented by ★. A possible escape route up to time t (with $N \geq 3$) is drawn with arrows.

RWDRE

- Let $I_x = (x - 1/2, x + 1/2)$.
- If $\eta_n^\lambda(I_{X_n}) > 0$ then $\begin{cases} X_{n+1} = X_n + 1 \text{ with probab. } p_\bullet \\ X_{n+1} = X_n - 1 \text{ with probab. } 1 - p_\bullet \end{cases}$.
- If $\eta_n^\lambda(I_{X_n}) = 0$ then $\begin{cases} X_{n+1} = X_n + 1 \text{ with probab. } p_\circ \\ X_{n+1} = X_n - 1 \text{ with probab. } 1 - p_\circ \end{cases}$.
- Questions: LLN, CLT.

Space-Time Correlations

- Detection is essentially a percolation problem on the space-time environment generated by HPS.
- Random walk increments correlate through the space-time correlations of HPS.
- HPS exhibits strong correlations along characteristic lines.

Sprinkled Decoupling (Today's Talk)

- If f_1 and f_2 are increasing functions of η then

$$\mathbb{E}^{\lambda}[f_1]\,\mathbb{E}^{\lambda}[f_2] \leq \mathbb{E}^{\lambda}[f_1 f_2] \,.$$

- If the supports of f_1 and f_2 are far apart and $\lambda < \lambda'$ then

$$\mathbb{E}^{\lambda}[f_1 f_2] \leq \mathbb{E}^{\lambda'}[f_1]\,\mathbb{E}^{\lambda'}[f_2] + \text{ small error } \,.$$

- Control of distance and error depends on $\lambda' - \lambda$.

Sprinkled Decoupling

Play a key role in multiscale renormalization schemes for strongly correlated systems:

- Sidoravicius and Sznitman. Percolation for the vacant set of random interlacements. *Comm. Pure Appl. Math.* (2009).

- Sznitman. Vacant set of random interlacements and percolation. *Ann. Math.* (2010).

Sprinkled Decoupling

More recently, in the context of particle systems:

- Hilário, den Hollander, Sidoravicius, dos Santos and Teixeira. Random walk on random walks. *Electron. J. Probab.* (2015).
- Baldasso and Texeira. How can a clairvoyant particle escape the exclusion process? *Ann. IHP Probab. Statist.* (2018).
- Hilário, Kious and Texeira. Random walk on the simple symmetric exclusion process. *Commun. Math. Phys.* (2020).

Back to HPS: Hydrodynamic Limit

Aldous and Diaconis heuristics:

- $w(x,t) =$ spatial process around x at time t.
- $\partial_t w(x,t) = d(x,t)$, where $d(x,t) =$ distance from x to x^*.
- $w(x,t)$ approximates PPP with rate $u(x,t) = \partial_x w(x,t)$.
- $\partial_t w(x,t) = d(x,t) = \frac{1}{u(x,t)} = \frac{1}{\partial_x w(x,t)}$.
- Burges' equation: $\partial_t u + \partial_x f(u) = 0$, where $f(u) = -\frac{1}{u}$.

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Slow Mixing Along Characteristics

- Method of characteristics:

$$u(x_0 + tf'(u(x_0)), t) = u(x_0)\,.$$

A disturbance made in the initial data travels along characteristic lines. In our case $f'(u) = \frac{1}{u^2}$.

- Stationary regime $u(x) = \lambda \implies f'(u(x)) = \frac{1}{\lambda^2}$.

- HPS exhibits slow mixing along characteristic lines:

$$\mathrm{Cov}^{\lambda}\left[\eta_0(x)\eta_t(x + t/\lambda^2)\right] \sim t^{-2/3} \text{ (KPZ scaling)}\,.$$

Sprinkled Decoupling Inequality for HPS

- $\mathcal{M}$ locally finite counting measure on $\mathbb{R}$.
- $\eta : [0, \infty) \to \mathcal{M}$ cadlag trajectory.
- $\eta \leq \bar{\eta}$ if $\forall\ t \geq 0\ \forall\ A \subseteq \mathbb{R}$ we have that $\eta_t(A) \leq \bar{\eta}_t(A)$.
- $f_1(\eta), f_2(\eta) \in [0, 1]$ for all η, support within space-time boxes B_1 and B_2, respectively.
- Sprinkling parameter $\epsilon = \left|\lambda^{-2} - \lambda'^{-2}\right|$.
- c_i uniform for all $\lambda, \lambda' \in K$, $K \subseteq (0, \infty)$ compact.

Sprinkled Decoupling Inequality for HPS

Theorem [P. and Viveros]

There exist $c_1, c_2 > 0$ such that for all

$$d \overset{def}{=} \mathrm{d}(B_1, B_2) > c_1 \epsilon^{-1} \big(\mathrm{per}(B_1) + \mathrm{per}(B_2) \big) \, ,$$

if $\lambda < \lambda'$ and f_i increasing then

$$\mathbb{E}^{\lambda} [f_1 f_2] \leq \mathbb{E}^{\lambda} [f_1] \, \mathbb{E}^{\lambda'} [f_2] + 10 e^{-c_2 \epsilon^4 d} \, ,$$

if $\lambda' < \lambda$ and f_i decreasing then

$$\mathbb{E}^{\lambda} [f_1 f_2] \leq \mathbb{E}^{\lambda} [f_1] \, \mathbb{E}^{\lambda'} [f_2] + 10 e^{-c_2 \epsilon^4 d} \, .$$

Proof: Lateral Decoupling + Domination

Write $d = \mathrm{d}(B_1, B_2) = d_h + d_v$.

Lemma - Lateral Decoupling

There exist $c_3, c_4 > 0$ such that if

$$d_h \geq c_3\Big(d_v + \mathrm{per}(B_1) + \mathrm{per}(B_2)\Big),$$

then

$$\Big|\mathbb{E}^{\lambda}[f_1 f_2] - \mathbb{E}^{\lambda}[f_1]\,\mathbb{E}^{\lambda}[f_2]\Big| \leq 4e^{-c_4 d_h} \leq 4e^{-\frac{c_4}{1+c_3^{-1}}d}.$$

Proof: Lateral Decoupling + Domination

Lemma - Domination

Consider the basic coupling $(\eta^\lambda, \eta^{\lambda'})$, with η_0^λ independent of $\eta_0^{\lambda'}$. There exists $c_5 > 0$ such that

$$\mathbb{P}\left[\forall\, u \in [t, t+s]\,, \forall\, A \subseteq (a, b]\,, \eta_u^\lambda(A) \leq \eta_u^{\lambda'}(A)\right] \geq 1 - 10e^{-c_5 \epsilon^4 t}\,,$$

for all $t > 4\epsilon^{-1}\left(b - a + s\lambda^{-2}\right)$.

HPS and Last-Passage-Percolation

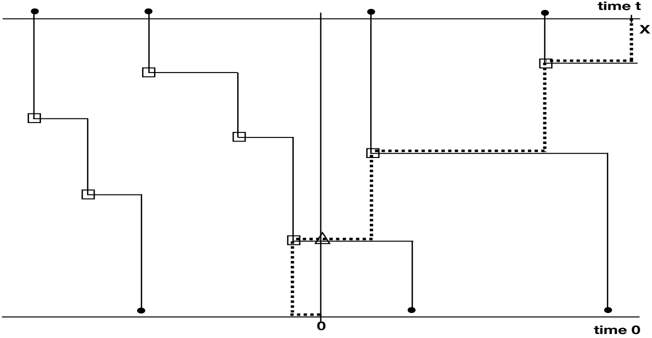

Figure: In this picture, $L_t^\lambda(x) = 3$, $L_t^\lambda(0) = 1$ and $\eta_t^\lambda((0,x]) = 2$. A maximal increasing path through $\square$-Poisson (clocks) from $(0,0)$ to (x,t) is drawn with dotted broken lines.

HPS and Last-Passage-Percolation

Define

$$L\left((z,s),(x,s+t)\right) = \text{ maximal increasing } \square\text{-Poisson path}\,.$$

and

$$M^{\lambda}(z) = \begin{cases} \eta_0^{\lambda}((0,z]) & \text{for } z>0 \\ -\eta_0^{\lambda}((z,0]) & \text{for } z \le 0\,. \end{cases}$$

The flux of particles $L_t^{\lambda}(x)$ can be written as

$$L_t^{\lambda}(x) = \sup\left\{ M_0^{\lambda}(z) + L((z,0),(x,t)) \,:\, z \in (-\infty, x] \right\}\,,$$

and

$$\eta_t^{\lambda}((a,b]) = L_t^{\lambda}(b) - L_t^{\lambda}(a)\,.$$

Proof of Domination

Lemma - Domination via Exit-Points

Let

$$Z_t^{\lambda}(x) = \sup\left\{z \in (-\infty, x] \,:\, L_t^{\lambda}(x) = M^{\lambda}(z) + L((z,0),(x,t))\right\} .$$

If $Z_t^{\lambda}(b) \leq Z_t^{\lambda'}(a)$ then

$$L_t^{\lambda}(y) - L_t^{\lambda}(x) \leq L_t^{\lambda'}(y) - L_t^{\lambda'}(x) \,,$$

for all $x, y \in [a, b]$ with $x < y$.

Proof of Domination

Lemma - Symmetries

$$Z_t^\lambda(x+h) \overset{dist}{=} Z_t(x)+h \text{ and } Z_t^\lambda(x) \overset{dist}{=} \lambda Z^1_{t/\lambda}(\lambda x)\,.$$

Lemma - Large Deviations

There exists an universal constant $c_0 > 0$ such that

$$\max\left\{\mathbb{P}\left[Z_t^1(t) > \epsilon t\right]\,,\mathbb{P}\left[Z_t^1(t) < -\epsilon t\right]\right\} \leq 5e^{-c_0\epsilon^4 t}\,,$$

for all $t > 0$.

Remark

Exit-points follow characteristic lines backward in time.

Proof of Domination

By monotonicity and Domination via Exit Points, it suffices to prove that

$$\mathbb{P}\left[Z_t^{\lambda}(b) \leq Z_{t+s}^{\lambda'}(a)\right] \geq 1 - 10e^{-c_5\epsilon^4 t}\,,$$

for all $t > 4\epsilon^{-1}\left(b - a + s\lambda^{-2}\right)$, which follows by combining Symmetries together with Large Deviations.

Final Remarks

- The key contribution is Domination.
- The same method (via exit points) can be use to prove sprinkled decoupling for stationary exponential last-passage percolation (SELPP).
- Is it possible to relate sprinkled decoupling for SELPP with sprinkled decoupling for TASEP?